Linear Approximation and Applications

1 Introduction

In this module we discuss a linear approximation method. That also includes an equation of a tangent line and differentials. As in the Optimization Module, in our presentation we avoid technicalities allowing students the opportunity to discover and explore those methods intuitively.

2 Linear Approximation

Example 1. Let us consider the following situation. You are driving a car on a highway. Your friend sitting next to you asks when will you arrive at the next rest area. At that moment you are passing the sign showing that the nearest rest area is 2 miles away. You look on the speedometer showing speed 60 mi/hour and tell your friend that you will arrive at the rest area in about 2 min.

Does your answer make sense? Are you able to predict how much time will it take to reach the rest area? You only know your speed at that moment. Your speed may be increased by you or slowed down by traffic. And, after all, nobody can drive a car with an exact constant speed.

Let us analyze your reasoning. First of all, you understand that there is no way you can know the exact time of arrival at the rest area. But, at the same time, your friend does not need that exact time either. What he needs is a good approximation of the arrival time. When you saw 60 mi/hour on your speedometer you made your prediction based on this speed because you understood that 2 minutes is a short time, so in 2 minutes your speed would not change significantly. Therefore you can assume that during the next two minutes the speed of your car will be close to 60 mi/hour. So the time to the rest area equals $60\, mi/hour \times 2/60\, hours = 2\, mi$.

Example 2. How many vehicles are registered in the United States in 2016? The data from US Department of Transportation shows that the number of vehicles registered in US was approximately 255,876,820 in 2013, and 260,350,940 in 2014. So in 2014 the rate of change of the number of registered vehicles was 4,474,120 per year. To estimate the number of vehicles in 2016 we assume that in 2015 and in 2016 the rate is same as in 2014. Therefore, our prediction is that in 2016 the number of vehicles is 260,350,940+2×4,474,120=269,299,180.

These examples show the main idea of a linear approximation. Assume that we know the values of a function $f(x)$ and its derivative $f'(x)$ at point a and we are looking for the value of $f(x)$ at some point x that is close to a. Of course what we know is insufficient to find the exact value $f(x)$. But in many real life problems instead of this exact value, the approximation can be used. So we come up with the following problem: given $f(a)$ and $f'(a)$ find an approximate value of f at $x = a + h$. When we use a linear approximation, we assume that since x is close to a the rate of change $f'(x)$ does not change too much in the interval $[a, x]$, as in Example 1 the speed of car does not considerably change in

2 min. Thus in a small interval $[a, x]$ $f'(x)$ can be considered as close enough to $f'(a)$. Therefore,

$$f(x) \approx f(a) + f'(a)(x - a). \tag{1}$$

The idea of a linear approximation, as simple as it is, may be a very efficient tool and the following exercise demonstrates it.

Exercise. In demographics, the world population is the total number of humans currently living. In March 2014 the world population was about 7 billion. As of March 2016, it was estimated at 7.4 billion. Use linear approximation to make prognosis about the world population in March of 2017, March of 2019.

3 Differential

The concept of a differential is very close to the one of a linear approximation. Let us rewrite formula (1) in the following form:

$$f(x) - f(a) \approx f'(a)(x - a). \tag{2}$$

Let us slightly change notations in this formula replacing a by x and x by $x+\Delta x$:

$$f(x + \Delta x) - f(x) \approx f'(x)\Delta x. \tag{3}$$

The expression in the left side of formula (3) represents the change of $f(x)$ occurring when x changes by Δx. Let Δf denotes the change of $f(x)$:

$$\Delta f := f(x + \Delta x) - f(x), \tag{4}$$

then formula (3) can be rewritten in the following form:

$$\Delta f \approx f'(x)\Delta x. \tag{5}$$

This remarkable formula has a very simple meaning. It says that the change of function is approximately proportional to the change of the variable Δx. For example, if a car moves along highway and its speed at some moment of time equals $v\ m/sec$ then the distance covered in Δt seconds can be approximated by $v\Delta t\ m$.

In Calculus, the change Δx of an independent variable x is denoted by dx and is called a differential of an independent variable x. The expression $f'(x)dx$ is denoted by df and is called the differential of a function f. Thus,

$$\Delta f \approx df := f'(x)\Delta x. \tag{6}$$

Remark. There is an essential difference between differentials of an independent variable and a function. While the differential of an independent variable is exactly the change of that variable, the differential of a function is only an approximation of the change of that function.

Exercise. Can you find examples of functions such that their changes equal the differentials precisely?

It is clear from what we learned about differentials that they would be useful when one is interested in approximating the impact of a change in some quantities.

Example. A company is selling cell phones and the monthly demand x is the following function of the price \$$p$:

$$x = f(p) := 300,000 - 400p + 0.2p^2. \tag{7}$$

The current price is \$300. How much will demand drop if the price is increased by 10 cents?

First of all we understand that formula (7) is not exact, it gives a reasonable approximation only. Therefore, it is not possible to find an exact value of the change of demand and we need an approximation only. Thus it makes sense to use a differential as an approximation of the change of demand. Also in this case, price p is an independent variable, since the price increased by 10 cents, $dp = 0.1$.

The demand x is a dependent variable, thus,

$$\Delta x \approx f'(p)dp.$$

From (7),

$$f'(p) = -400 + 0.4p.$$

Therefore

$$\Delta x \approx f'(100) \cdot 0.1 = (-400 + 0.4 \cdot 300) \cdot 0.1 = -28.$$

So the monthly demand would decrease by 28 cellphones.

Exercise. Given the price demand equation $x = 50,000 - 100p - 10p^2 + 0.1p^3$, use differential to find the change of the demand if the price is

a) increased from \$10 to \$11,

b) decreased from \$20 to \$19.70.

4 An Equation of a Tangent Line.

We have already seen the close relationship between concepts of a linear approximation and a differential. Another concept closely related to a linear approximation is the tangent line. A tangent line to the graph of a function is a graph of a linear approximation of a function at that point.

Since the linear approximation of $f(x)$ at point a is

$$L_a(x) = f'(a)(x - a) + f(a),$$

then the tangent line to the graph of $y = f(x)$ at $(a, f(a)$ is the graph of $y = L_a(x)$:

$$y = f'(a)(x - a) + f(a). \tag{8}$$

Example. A military plane takes off from a military base. Its trajectory is a parabolic curve $y = 2000x - x^2$. At the point with coordinates $(1200, 960000)$ the plane launches a missile towards the target with the coordinates $(1800, 720000)$. The path of the missile is a straight line tangent to the trajectory of the plane at the point of the launch. Will missile hit the target?

Solution. The trajectory of a missile is a tangent line to the graph of $y = 2000x - x^2$ at $(1200, 960000)$. So to find the equation of the tangent line we have to find the derivative of $y = 2000x - x^2$, that is

$$y'(x) = 2000 - 2x.$$

Then the equation of a tangent line is

$$y = y'(a)(x - a) + y(a).$$

Since $y(1200) = 2000 \cdot 1200 - 1200^2 = 960000$ and $y'(1200) = 2000 - 2 \cdot 1200 = -400$, the equation of a tangent line is

$$y = -400(x - 1200) + 9600.$$

Now we have to substitute the coordinates of the target $(1800, 720000)$ in the equation of the tangent line to see whether the target is located on the trajectory of the missile. Since

$$-400(1800 - 1200) + 9600 = 720000,$$

we can congratulate the pilot with an excellent launch of the missile.

5 Mini–projects

Mini–project 1. Linear approximation. The sun rises at different times, depending on the date and location. Record the sunrise at the city of your residence on Aug 1, Aug 3, Aug 4, Aug 7 and then estimate the sunrise time for Aug 18th.

Mini–project 2. Device Tolerance. Ohm's law states that the current through a conductor between two points is directly proportional to the voltage across the two points. Introducing the constant of proportionality, the resistance, one arrives at the usual mathematical equation that describes this relationship:

$$I = \frac{V}{R},$$

where I is the current through the conductor in units of amperes, V is the voltage measured across the conductor in units of volts, and R is the resistance of the conductor in units of ohms. Electrical sockets (outlets) in the United States of America usually supply electricity at between 110 and 120 volts. Assume that the resistance of the device plugged into an outlet is known. Let the voltage be 120 volts and the resistance be 100 ohms. Then one can use Ohm's Law to find the current. However in real world applications the voltage is always slightly different from its nominal value and the resistance of the device is not a precise number but is given with some accuracy. Thus an important problem is to estimate how much the errors in the voltage and in the resistance will affect the current. The most convenient way to address this question is to use the differential.

i) Find the current passing through the device if $V = 120$ volts and $R = 100$ ohms;

ii) Use differential to find how much the current would change if $R = 100$ ohms but the voltage dropped by 3 volts;

iii) Use differential to find how much the current would change if $V = 120$ volts but the actual resistance of the device is 102 ohms;

iv) Use differential to find how much the current would change if $V = 120$ volts but the actual resistance of the device differs from the assigned value of 100 ohms by 1.5

v) Assuming that $V = 120$ volts and that the current must be in the range from 1.15 to 1.25 amperes, use the differential to estimate how much the actual resistance may vary from its nominal value of 100 ohms;

vi) In electrical engineering class of accuracy is a figure which represents the error tolerance of a device. Using parts ii -v of your project write a short essay explaining how differential can be used in the tolerance analysis.

Mini–project 3. Baseball. (Stu Schwartz, www.MasterMathMentor.com)

On your desk, there is a standard baseball, a piece of string, and a ruler.

Do the following:

1) To the best of your ability, find the circumference of the ball in inches. $C =$ ________________.

2) How much error will you build in to your answer above that is, how far off will you allow yourself to be. Answer in inches. $\Delta C = dC =$ ________________. Be careful if you use a value that is too small, you are saying that you are very sure of your circumference above (opening yourself up to a lawsuit). If you use a value that is too large, you are saying that you are not real confident of your value which does not instill much confidence.

3) So what is your radius in inches? 5 decimal places. $r =$ ________________. You may wish to store this variable on your calculator. Call it R.

4) So what is your maximum error in the radius? $\Delta r = dr =$ ________________. You may wish to store this variable on your calculator. Call it D.

5) What do you believe the volume of the ball is (5 decimal places)? $V =$ ________________.

6) If we were to find the volume of the ball (volume of a sphere is $4\pi r^3/3$), what is the greatest error (5 decimal places) that you could have made based on your values of r and Δr? What variable are you finding? ________________.

7) Now approximate the greatest error (5 decimal places) you could have made in finding the volume using your values of r and r. What variable are

you finding? ________________.

8) What is the difference between 6) and 7). What variable are you finding? ________________.

9) Calculate the percentage error in your approximation. ________________.

10) If we were to wrap the ball in wrapping paper, to 5 decimal places, what is the greatest error (5 decimal places) that you could have made based on your values of r and Δr? Surface area of a sphere is $4r^2$. What variable are you finding? ________________.

11) Now approximate the greatest error (5 decimal places) you could have made in finding the surface area using your values of r and Δr. What variable are you finding? ________________.

12) What is the difference between 10) and 11) ________________? What variable are you finding? ________________. 13) Calculate the percentage error in your approximation. ________________.

14) Do you believe that all baseballs are made with the exact specifications? Explain.

Mini–project 4. Trip to Mars. In this mini–project you will plan a trip from the Earth to the orbit of the Mars. The orbit of the Earth is an ellipse with the Sun located at the focus. Let us consider the plane of the Earth's orbit and denote by x and y the coordinates of the Earth. The equation of the Earth's orbit can be written as

$$x^2 + \frac{y^2}{0.9999^2} = 1,$$

where x and y are measured in astronomical units AU (1 AU=149597870 km.).

For the sake of simplicity we assume that the orbit of the Mars is in the same plane and is given by the equation

$$\frac{x^2}{1.5237^2} + \frac{y^2}{1.5170^2} = 1$$

To reach the orbit of the Mars we will use the orbital velocity of the Earth $v =$ 30 km/s. We assume that the spaceship launched at some point of the orbit moves along the straight line that is tangent to the orbit with the velocity equal to the orbital velocity of the Earth. The goal of this mini–project is to find a point at which the spaceship intersects the orbit of the Mars and to calculate how much time it will take.

Plan of work:

i) choose several points on the Earth's orbit;

ii) for each point find an equation of the line tangent to the orbit (you may need an implicit differentiation);

iii) for each tangent line find the point of intersection with the orbit of Mars;

iv) for each tangent line find the distance from one orbit to the other traveled by the spaceship;

v) for each tangent line find the time of the travel from one orbit to the other.