Optimization Problems

1 Introduction

In this module we discuss optimization problems, their applications and methods of solution. In our presentation we avoid technicalities providing the students the opportunity to discover and explore those methods intuitively. Traditionally mathematics is taught in a logical sequence starting with definitions and theorems followed by problems. What is lost is the challenge and reward of discovery of mathematical concepts. Indeed, instead of concentrating on mathematical techniques, it requires some additional reasoning to facilitate the learning process.

2 Optimization Problems

This module is devoted to solving a category of problems, called optimization, which is the problem of finding the maxima and minima of a given function. This problem is one of the most important areas investigated in mathematics and the most important problem in applications. Below we discuss some applications. Let us start with one of them.

Assume that you work for a company that produces some goods, for example it manufactures and sells cell phones. You work in the pricing department and you are responsible to determine the price of the product. Of course your goal is to choose the right price that maximizes the revenue. What should you do to get the best result? We will show you how Calculus can help you to answer this question.

Calculus deals with functions, equations, graphs, and other mathematical concepts. In order to use Calculus, first you must translate the problem into the language of mathematics, reformulate the problem as a mathematical problem. This process is called Mathematical Modeling. It requires skills in both mathematics and business.

So, what is the right price? You may want to make the price higher but how many customers will buy your phones at a high price? If you decrease the price you will increase the customer demand, but would it maximize the revenue? To answer these questions, we will build a mathematical model based on the relationship between the price and the demand. Hopefully the marketing department of your company can provide you with this relationship expressed in terms of the following price-demand equation:

$$d = f(p). \tag{1}$$

This equation shows the dependence of demand d on price p. In mathematics such dependence is called a function, that is, d is a function of p. Since the values of d and p can vary they are called variables. In (1) d depends on p, thus d is said to be a dependent variable, and p is an independent variable. Correspondingly, f that represents this relationship is called a function. Functions are what

Calculus is about. In other words, Calculus studies the dependence between different variables, or simply, Calculus studies functions.

Example 1. Well, we got a call from the marketing department. They came up with the following price-demand equation:

$$d = 10,000 - 8p, \quad \text{that is} \quad f(p) = 10,000 - 8p. \tag{2}$$

Therefore, for the given price p after selling d phones the revenue will be:

$$R = pd. \tag{3}$$

After substituting $d = 10,000 - 8p$ in this equation we express R as a function of p:

$$R(p) = p(10,000 - 8p). \tag{4}$$

The price cannot be negative, so we assume that $p \geq 0$. You may argue that from the business point of view $p = 0$ does not make sense either, but mathematically it is possible that phones are free. The demand d cannot be negative as well, so $d \geq 0$. That brings the following inequality,

$$d = p(10,000 - 8p) \geq 0. \tag{5}$$

Let us solve the problem assuming that $p \geq 0$. Obviously, $p = 0$ does satisfy the inequality. Let us consider $p > 0$ and multiply both sides of (5) by a positive number $1/p$. The multiplication by a positive number does not change the sign in the inequality and we obtain that

$$10,000 - 8p \geq 0 \tag{6}$$

or

$$p \leq \frac{10,000}{8} = 1250. \tag{7}$$

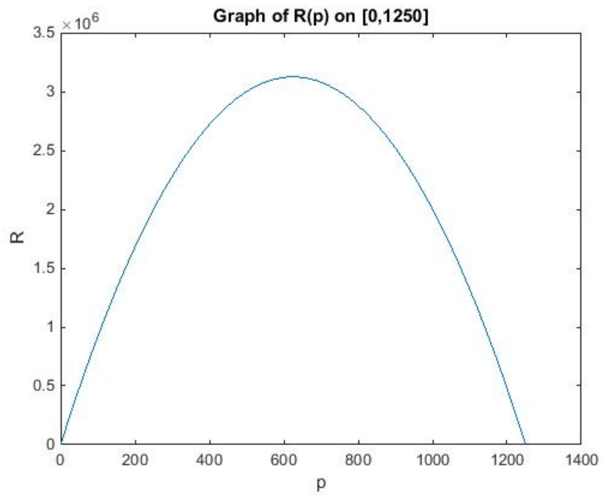
Now we can consider function $R(p)$ for $0 \leq p \leq 1250$ only. Thus, we will consider $R(p)$ in the closed interval $[0, 1250]$.

Our main task can be summarized as the following mathematical problem.

Optimization problem. Find a value of p in the interval $[0, 1250]$ for which the value of $R(p)$ is maximal.

To solve this problem let us take a look at the graph of $R(p)$ in the interval $[0, 1250]$ presented in Figure 1. If we choose $p = 0$ we obviously get zero revenue. When we increase price p from zero, the revenue first increases, but after some value p^* of p it decreases again and becomes zero at $p = 1250$. Clearly at $p = p^*$ one gets the maximum revenue $R(p^*)$. So p^* is the solution of the optimization problem we are looking for.

Now the big question is how p^* can be found. The main observation about p^* is that it separates the interval where the function increases from the interval where it decreases. Thus in order to find p^* we have to obtain a tool for finding such intervals. The **great idea of Calculus** is that this can be done by analyzing **the instantaneous rate of change** of a function. Though we



deal with it every day, the concept of an instantaneous rate of change is not as simple as it seems. Indeed, looking at the speedometer of your car you can see the speed with which your car moves. What does your speedometer show? What does speed mean? You may say that speed is rate of change of distance with respect to time, but is it an average or an instantaneous rate of change? Think about it!

Now, first let us introduce the notion of an average rate of change for $R(p)$. Let us assume that price p changes from $p = p_1$ to $p = p_2$ and that $p_2 > p_1$. How much does the revenue change? Obviously the revenue changes from $R(p_1)$ to $R(p_2)$. Therefore, the change of the revenue is $R(p_2) - R(p_1)$. So the increase of the price by $p_2 - p_1$ dollars generates the change of the revenue by $R(p_2) - R(p_1)$ dollars. Now the average change of the revenue per 1 dollar is

$$\frac{R(p_2) - R(p_1)}{p_2 - p_1}.$$

So when the revenue increases, that is $R(p_2) > R(p_1)$, the above average rate of change is positive, and when the revenue decreases it would be negative. This answers our big question but only yields an average.

To get a more precise answer, let us take a closer look at the rate of change of the revenue when the price is close to some fixed value like p. Assume that the price is changed by Δp dollars, that is from p to $p + \Delta p$. Then the revenue changes from $R(p)$ to $R(p + \Delta p)$. The corresponding average rate of change is

$$\frac{R(p + \Delta p) - R(p)}{\Delta p}.$$

Since $R(p) = p(10,000 - 8p) = 10,000p - 8p^2$ this rate equals to

$$\frac{(10,000(p + \Delta p) - 8(p + \Delta p)^2) - (10,000p - 8p^2)}{\Delta p}$$

$$= \frac{10,000p + 10,000\Delta p - 8(p^2 + 2p\Delta p + (\Delta p)^2) - 10,000p + 8p^2}{\Delta p}$$

$$= \frac{10,000p + 10,000\Delta p - 8p^2 - 16p\Delta p - 8(\Delta p)^2 - 10,000p + 8p^2}{\Delta p}$$

$$= \frac{10,000\Delta p - 16p\Delta p - 8(\Delta p)^2}{\Delta p} = 10,000 - 16p - 8\Delta p.$$

As you can see in this formula, the average rate of change of the revenue when price changes from p to $p + \Delta p$ depends not only on the price p, but also on its change Δp. However, our goal is to determine whether the revenue is increasing or decreasing when the price is close to p. To do this, we need to find an average rate of change when the value of the price is close to p, or in other words, we have to take small Δp. Thus let us assume that Δp becomes smaller and smaller, or in other words, let us assume that Δp tends to zero. Then the average rate of change $10,000 - 16p - 8\Delta p$ approaches $10,000 - 16p$. This value is called an instantaneous rate of change. It is approximately equal to the average rate of change corresponding to a very small Δp. Like the average rate of change, the instantaneous rate of change is positive if $R(p)$ is increasing in vicinity of point p, and is negative if $R(p)$ is decreasing.

Now, by using the instantaneous rate of change $10,000 - 16p$, we can figure out where in the interval $[0, 1250]$, the $R(p)$ is increasing, and where it is decreasing. The instantaneous rate of change is positive, when

$$10,000 - 16p > 0 \quad \text{or} \quad p > \frac{10,000}{16} = 625$$

and is negative, when

$$10,000 - 16p < 0 \quad \text{or} \quad p < \frac{10,000}{16} = 625.$$

This means that revenue increases when price increases from 0 to 625 and decreases when price increases from 625 to 1250. Thus our analysis shows that at $p = 625$ the revenue is maximal. Therefore, the right selling price is \$625.00.

Exercises.

1. The following table shows the estimated percentage of the population in Europe that use cell phones. (Midyear estimates are given.)

Year	1998	1999	2000	2001	2002	2003
P	28	39	55	68	77	83

Find the average rate of cell phone usage growth (what are its units?) (i) from 2000 to 2002 (ii) from 2000 to 2001 (iii) from 1999 to 2000

2. A Formula One car accelerates once started. The distance covered by the car is given as a function of time $x = 3t^2$, where time t is measured in seconds, and distance x is in meters. Find

(a) the average speed of the car between
(i) $t = 0$ and $t = 5$
(ii) $t = 5$ and $t = 10$
(iii) $t = 0$ and $t = 10$

(b) the instantaneous speed of the car at
(i) $t = 1$
(ii) $t = 5$
(iii) $t = 8$

3. It costs a monopolist \$5/unit to produce a product. If he produces x units of the product, then each can be sold for $10 - x$ dollars. To maximize the profit, how many units should the monopolist produce?

4. The monthly revenue R and production cost function C of a company are given as follows:

$$R = -0.007Q^2 + 32Q, \quad C = 0.004Q^2 + 2.2Q + 8,$$

where Q is the production size. The profit P is the difference between the revenue and the production cost, $P = R - C$. Find the quantity Q at which maximum profit will occur and the amount of the profit.

Mini–project 1. A pig weighing 200 pounds gains 5 pounds per day and costs \$0.60 per day to maintain. The market price for pigs is \$0.70 per pound, but is falling \$0.01 per day. When should the pig be sold?

3 Instantaneous Rate of Change

As you can see, the solution of the problem stated in the previous section was based on the concept of an instantaneous rate of change. Let us discuss this concept a little more. To find the instantaneous rate of change the following three-step procedure was used:

(i) the first step was to find the average rate:

$$\frac{R(p + \Delta p) - R(p)}{\Delta p} = \frac{10,000\Delta p - 16p\Delta p - 8(\Delta p)^2}{\Delta p}; \tag{8}$$

(ii) the goal of the second step was to cancel out Δp:

$$\frac{10,000\Delta p - 16p\Delta p - 8(\Delta p)^2}{\Delta p} = 10,000 - 16p - 8\Delta p; \tag{9}$$

(iii) the third step was to substitute 0 into $10,000 - 16p - 8\Delta p$.

Why do we need this procedure in order to define an instantaneous rate of change? Obviously, $\Delta p = 0$ cannot be substituted into the formula of the average rate of change (8) simply because the denominator of a fraction cannot be zero. The average rate represented in (8) is undefined for $\Delta p = 0$. The last step had become possible only after the second step was done. Let us take a closer look at the crucial step (ii). It claims that

$$\frac{10,000\Delta p - 16p\Delta p - 8(\Delta p)^2}{\Delta p} = 10,000 - 16p - 8\Delta p,$$

however, the left and the right hand side expressions are not identical, since they are equal only if $\Delta p \neq 0$. When $\Delta p = 0$ the left hand side expression does not make sense, but the right hand side expression does. To address this problem, we are looking for a value to which the left side expression approaches when $\Delta p \to 0$ and at the same time we require that $\Delta p \neq 0$. Therefore the difference between the left and the right hand side expressions makes no difference.

Now we can state the following definition.

Definition 1. The instantaneous rate of change of a function $R(p)$ at a point p is the value to which the average rate of change

$$\frac{R(p + \Delta p) - R(p)}{\Delta p}$$

approaches when $\Delta p \to 0$ but $\Delta p \neq 0$.

Example 2. The manager of a car dealership was informed that the demand of cars during the year can be described by means of the following formula:

$$d = -t^3 + 16.5t^2 - 54t + 90, \quad 0 \leq t \leq 12.$$

When is the demand maximal and when is it minimal?

To solve the problem we follow the procedure used in Example 1. That is we are looking for the instantaneous rate of change of the demand, d, with respect to time t. So let us apply again the three-step procedure described above.

(i) The first step evaluates the average rate of change of d when time changes from t to $t + \Delta t$. First, we find

$$d(t + \Delta t) = -(t + \Delta t)^3 + 16.5(t + \Delta t)^2 - 54(t + \Delta t) + 90$$

$$= -(t^3 + 3t^2\Delta t + 3t(\Delta t)^2 + (\Delta t)^3) + 16.5(t^2 + 2t\Delta t + (\Delta t)^2) - 54(t + \Delta t) + 90$$

$$= -t^3 - 3t^2\Delta t - 3t(\Delta t)^2 - (\Delta t)^3 + 16.5t^2 + 33t\Delta t + 16.5(\Delta t)^2 - 54t - 54\Delta t + 90.$$

Thus, the demand d change Δd is

$$\Delta d = d(t + \Delta t) - d(t)$$

$$= -t^3 - 3t^2\Delta t - 3t(\Delta t)^2 - (\Delta t)^3 + 16.5t^2 + 33t\Delta t + 16.5(\Delta t)^2 - 54t - 54\Delta t + 90$$
$$-(-t^3 + 16.5t^2 - 54t + 90)$$
$$= -3t^2\Delta t - 3t(\Delta t)^2 - (\Delta t)^3 + 33t\Delta t + 16.5(\Delta t)^2 - 54\Delta t.$$

Therefore, the average rate of change in the interval $[t, t + \Delta p]$ is

$$\frac{\Delta d}{\Delta t} = \frac{-3t^2\Delta t - 3t(\Delta t)^2 - (\Delta t)^3 + 33t\Delta t + 16.5(\Delta t)^2 - 54\Delta t}{\Delta t}. \quad (10)$$

(ii) As we discussed before, the expression (10) for the average rate of change is undefined for $\Delta p = 0$. In this step we replace it by the expression that equals to (10) for $\Delta p \neq 0$ but unlike (10) is defined for $\Delta p = 0$. To do this we have to cancel Δp in the following fraction:

$$\frac{\Delta d}{\Delta t} = \frac{\Delta t(-3t^2 - 3t\Delta t - (\Delta t)^2 + 33t + 16.5\Delta t - 54)}{\Delta t}$$
$$= -3t^2 - 3t\Delta t - (\Delta t)^2 + 33t + 16.5\Delta t - 54. \quad (11)$$

(iii) We are ready for the third step. We can substitute $\Delta t = 0$ in (11) and conclude that the instantaneous rate of change is

$$-3t^2 + 33t - 54. \quad (12)$$

Now we can use this instantaneous rate of change to find subintervals of $[0, 12]$ at which $d(t)$ either increases or decreases. $d(t)$ increases when the instantaneous rate of change is positive. That is

$$-3t^2 + 33t - 54 > 0.$$

To solve this inequality we factor the polynomial:

$$-3t^2 + 33t - 54 = -3(t^2 - 11t + 18) = -3(t - 2)(t - 9).$$

The inequality

$$-3(t - 2)(t - 9) > 0$$

implies $2 < t < 9$. Similarly,

$$-3(t - 2)(t - 9) < 0$$

implies $t < 2$ or $t > 9$.

Therefore $d(t)$ is increasing in $(2, 9)$ and decreasing in $[0, 2)$ and in $(9, 12]$.

At $t = 0$ the demand is $d(0) = 90$. From $t = 0$ to $t = 2$ the demand is decreasing from 90 to $d(2) = -2^3 + 16.5 \cdot 2^2 - 54 \cdot 2 + 90 = 40$. From $t = 2$ to $t = 9$ the demand is increasing from 40 to $d(9) = -9^3 + 16.5 \cdot 9^2 - 54 \cdot 9 + 90 = 211.5$. Finally, from $t = 9$ to $t = 12$ the demand is decreasing from 211.5 to $d(12) = -12^3 + 16.5 \cdot 12^2 - 54 \cdot 12 + 90 = 90$.

Thus, the minimal demand occurs at $t = 2$ and equals to 40, the maximal demand occurs at $t = 9$ and is equal to 211.5.

4 Limit

In previous sections a procedure for solving optimization problems based on the concept of an instantaneous rate of change was introduced. Starting from this section we introduce some mathematical terminology for this procedure. The most essential idea used above was that it may be possible to find the value that $\frac{R(p+\Delta p)-R(p)}{\Delta p}$ reaches to when Δp approaches zero, though $\frac{R(p+\Delta p)-R(p)}{\Delta p}$ is undefined for $\Delta p = 0$. In mathematics this phenomenon is called a **limit**.

Let $f(x)$ be a function defined in a vicinity of a point a, but not necessarily at the point a. It is said that the limit of $f(x)$ for $x \to a$ is a number, called L, if $f(x)$ becomes very close to L provided that x is sufficiently close to a but $x \neq a$. In this case we write

$$\lim_{x\to a} f(x) = L. \qquad (13)$$

Example. Find $\lim\limits_{x\to 1} \frac{x^2-2x+1}{x^2-1}$.

Since the denominator cannot be zero, $\frac{x^2-2x+1}{x^2-1}$ is undefined at $x = 1$. But according to the concept of limit it is not needed! While we are evaluating the limit (13) the value of function at $x = 1$ is irrelevant! (The function is not defined at $x = -1$ as well but it is also irrelevant since we assume that x is close to 1.)

Let us factor both numerator and denominator:

$$x^2 - 2x + 1 = (x-1)^2,$$

$$x^2 - 1 = (x-1)(x+1).$$

Therefore,

$$\frac{x^2-2x+1}{x^2-1} = \frac{(x-1)^2}{(x-1)(x+1)}.$$

After canceling out $x - 1$ we get that for every $x \neq 1$

$$\frac{x^2-2x+1}{x^2-1} = \frac{x-1}{x+1}. \qquad (14)$$

The functions in the left and right hand sides in (reflim1) are different since the left hand side function is not defined at $x = 1$, while the right hand side function is defined. However, since in the definition of the limit (13) the point $x = 1$ does not participate, we can use (14) for evaluation of the limit because for $x \neq 1$ (14) is satisfied. Thus,

$$\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{x+1}.$$

Exercises.

Evaluate the following limits:

i) $\lim\limits_{x\to 0} \frac{2x^2}{x^4+2x^3-6x^2}$

ii) $\lim\limits_{x\to 1} \frac{x^2-6x+5}{x-1}$

iii) $\lim\limits_{x\to -2} \frac{x^2-x-6}{x^2+x-2}$

iv) $\lim\limits_{x\to 2} \frac{x^3-2x^2}{x^4-x^3-2x^2}$

v) $\lim\limits_{x\to 0} \frac{x^3-2x^2}{x^4-x^3-2x^2}$

Mini–project 2. Concept of Limit. A student who just learned about limits asked you a question. He said that he thinks that the concept of the limit does not make any sense. His argument is this:

Consider the statement $\lim\limits_{x\to c} f(x) = L$. As it was explained in the class, this limit L does not depend on the value $f(c)$ of the function $f(x)$ at $x = c$. Moreover such a value may not even exist. Also, it was explained that the limit is a number to which $f(x)$ approaches when x becomes very close to c. So if we take a point a that is not equal to c, we are allowed to consider only points that are closer to c than a. So L does not depend on $f(a)$ either. So L depends neither on $f(c)$ nor on any $f(a)$ for $a \neq c$. But this means that L does not depend on f. So how can it happen that different functions have different limits?

Write an essay that answers this question.

5 Derivative

There is a special name derivative for an instantaneous rate of change. The notations for the derivative of a function $f(x)$ are $f'(x)$ and $\frac{df}{dx}$. The derivative is defined as a limit (compare with the definition of the instantaneous rate of change!):

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

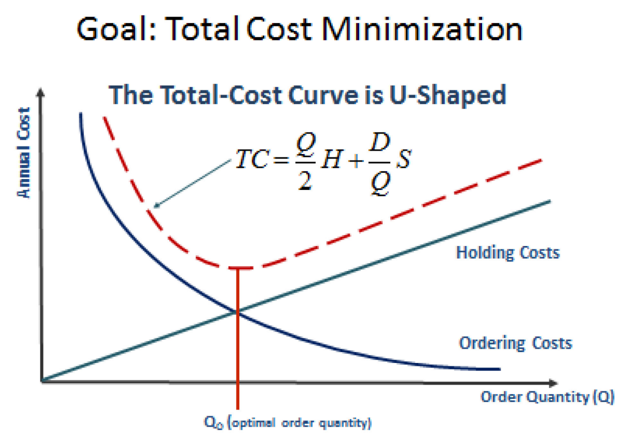
Mini–project 3. Total Annual Inventory Cost.
Total Cost=Annual Holding Cost+Annual Ordering Cost, that is

$$TC(Q) = \frac{Q}{2}H + \frac{D}{Q}S,$$

where
TC is the total cost.
Q is the order quantity in units,
H is the holding (carrying) cost per unit, usually per year,
D is the demand, usually in units per year,
S is the ordering cost per order.

The goal of the project is

a. to determine the optimal batch size (order quantity, Q) in order to minimize the total cost of the system (Fig. 2),



b. to find the economic order quantity assume this information: $D = 4,500$ units/ year, $S = \$36$, and $H = \$10$ per unit per year. To solve part **a** you have to find the value of Q at which the derivative (instantaneous rate of change) $\frac{d(TC)}{dQ}$ changes its sign from minus to plus.

The project 1 consists of the following

Steps:

1. Find $TC(Q + \Delta Q)$;

2. Find an average rate of change: $\frac{TC(Q+\Delta Q)-TC(Q)}{\Delta Q}$;

3. The average rate of change you found is undefined at $\Delta Q = 0$, since both the numerator and the denominator become zero. To find the instantaneous rate of change you have to find the limit of this ratio when $\Delta Q \to 0$. In order to be able to find the limit cancel ΔQ in both the numerator and the denominator of the ratio;

4. Find $\frac{d(TC)}{dQ} = \lim_{\Delta Q \to 0} \frac{TC(Q+\Delta Q)-TC(Q)}{\Delta Q}$

5. Find the value of Q at which $\frac{d(TC)}{dQ}$ changes its sign by solving the equation $\frac{d(TC)}{dQ} = 0$;

6. Explain why this value minimizes the total cost;

7. Apply part **a** to solve the part **b**.

Mini–project 4. Cost Minimization. (You may follow the steps of the previous project.) We want to keep paper towels on hand at all times. The cost of stockpiling towels is $\$h$ per case per week, including the cost of space, the cost of tying up capital in inventory, etc. The time and paperwork involved in ordering towels costs $\$K$ per order (this is in addition to the cost of the towels themselves, which is not relevant here because we are supposing that there are no price discount for large orders.) Towels are used at a rate of d cases per week. What quantity Q should we order at a time, to minimize cost?

Mini–project 5. Velocity of the ball. An engineering professor asks a student to drop a ball from a height of $y = 1.0\,m$ to find the time when it impacts the ground. Using a high-resolution stopwatch, the student measures the time at impact as $t = 0.452s$. The professor then poses the following questions:

(a) What is the average velocity of the ball?

(b) What is the speed of the ball at impact?

(c) How fast is the ball accelerating?

After answering the first question the student told the professor that he needs more information to answer second and third questions. So they went to the laboratory and used an ultrasonic motion detector again. The collected data they included in the following table:

Table 1. Additional data collected from the dropped ball.

t (seconds)	0	0.1	0.2	0.3	0.4	0.452
y(t) (meters)	1.0	0.951	0.804	0.559	0.215	0

Using this table the student calculated the average velocity $\bar{v} = \Delta y/\Delta t$ in each interval and filled the following table:

Table 2. Average velocity of the ball in different intervals.

Interval	[0,0.1]	[0.1,0.2]	[0.2,0.3]	[0.3,0.4]	[0.4,0.452]
$\bar{v}$ (meters/second)					

Based on this table the student proposed an approximate answers to the questions b and c, and claimed that an infinite number of measurements are needed to find the precise answer. The professor agreed but said that since an infinite number of measurements is not possible, they will choose an alternative approach. He suggested a quadratic curve $y(t) = 1 - 4.905t^2$ that fits data collected in the table 1. Using this function student was able to answer all three questions.

In this project you have to repeat the work of the student, to answer questions, a, b, and c, and to find an approximate answers to the question b and c that student found using table 2.