# Related Rates

## 1 Introduction

Related rates problems appear when there are two or more quantities related to each other by equations and one is interested in relations between their rates of change. To understand this let us start with an example when rates of change are constant.

**Example 1. Apple farm.**
Let us consider an apple farm with $x$ apple trees and yearly harvest $H$. Let us assume that the yearly harvest from every apple tree is 50 lb. So with every tree added the harvest increases by 50 lb, or, in other words, the rate of change of the harvest with respect to the number of trees is 50 lb per tree, that is $dH/dx = 50$. Now assume that the number of trees is increasing by 10 trees per year, thus, the rate of change of the number of trees with respect to time is 10 trees per year, that is $dx/dt = 10$. Now let us find the rate of change of the harvest with respect to time, that is $dH/dt$.

Let $H(x)$ denote harvest as a function of number of trees, and $x(t)$ denotes number of trees as a function of time $t$. The dependence of the harvest on time is given by $H(x(t))$ that is the composition of functions $H(x)$ and $x(t)$.

Then change of harvest per year equals 50 (change of harvest per one tree) multiplied by 10 (number of trees added in one year), or, in other words,

$$\frac{dH}{dt} = \frac{dH}{dx} \cdot \frac{dx}{dt}. \tag{1}$$

In Calculus this formula is called the Chain Rule. In this example, for the sake of simplicity, we assumed that all rates of change are constant. It is proved in Calculus that the same formula works for variable rates as well. Thus, let $t$, $x$, $y$ be some variables related to each other by formulas:

$$x = f(t), \quad y = g(x). \tag{2}$$

Then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}, \tag{3}$$

or, in more detailed notations,

$$[g(f(t))]' = g'(f(t))f'(t). \tag{4}$$

## 2 Exercises

**1. Effect of rainfall on predators and prey.** The population of lions, $L$, in Africa greatly depends on the population of their prey, $P$, which consists mainly of large mammals with a preference for zebras, impalas, wildebeest, buffalo, and warthogs. The extension of vegetation, $V$ , affects the population of prey, which feed on the vegetation, and the vegetation is affected by the amount of rainfall,

$r$. These dependencies can be as $L = f(P)$, $P = g(V)$, and $V = h(r)$. Data was collected on an African wildlife preserve and the functions $f$, $g$, $h$ were approximated as follows:

$$L = f(P) = 12P^2, \quad P = g(V) = 2V, \quad V = h(r) = r^{1/2}.$$

A change in the abundance of rainfall will affect the amount of vegetation. This change in vegetation will affect the population of prey which in turn will affect the population of lions on the reserve. Therefore, the amount of rainfall affects the lion population. Compute the rate of change in the population of lions with respect to the rainfall, $dL/dr$.

**2. Carbon monoxide level.** An environmental study of a certain community suggests that the average daily level of carbon monoxide in the air may be modeled by the formula

$$C(p) = (0.5p^2 + 17)^{1/2}$$

parts per million when the population is $p$ thousand. It is estimated that t years from now, the population of the community will be

$$p(t) = 3.1 + 0.1t^2$$

thousand. At what rate will the carbon monoxide level be changing with respect to time 3 years from now?3.

**3. Manufacturing cost.** At a certain factory, the total cost of manufacturing $q$ units is $C(q) = 0.2q^2 + q + 900$ dollars. It has been determined that approximately $q(t) = t^2 + 100t$ units are manufactured during the first $t$ hours of a production run. Compute the rate at which the total manufacturing cost is changing with respect to time 1 hour after production commences.

**4. Depreciation.** The value $V$ (in thousands of dollars) of an industrial machine is modeled by

$$V(N) = \left(\frac{3N + 430}{N + 1}\right)^{2/3},$$

where $N$ is the number of hours the machine is used each day. Suppose further that usage varies with time in such a way that

$$N(t) = \sqrt{t^2 - 10t + 45},$$

where $t$ is the number of months the machine has been in operation.

a. How many hours per day will the machine be used 9 months from now? What will be the value of the machine at this time?

b. At what rate is the value of the machine changing with respect to time 9 months from now? Will the value be increasing or decreasing at this time?

## 3 Mini–projects

**Mini–project 1. Hot air balloon.**

(http://i-want-to-study-engineering.org/q/hot_air_balloon/)

In this project you will deal with a hot air balloon carrying 5 people, 80 kg each. The volume of the balloon is 3000 $m^3$ and total mass of the balloon itself and of the passenger cabin is 400 kg. On the sea level the balloon was filled with hot air with the absolute temperature $T$. The outside temperature at sea level was $288^o K$ and outside atmospheric pressure at sea level was 101.33 kPa. These conditions allowed air balloon with these 5 passengers to fly horizontally on the sea level. The goal of this project is to control the temperature of the balloon that allows it to fly on the required altitude.

You can assume that the hot air behaves as an ideal gas that is governed by the following equation:

$$pV = nRT, \tag{5}$$

where $p$ is the pressure of the gas, $V$ is the volume of the gas, $n$ is the amount of substance of gas (in moles), $R$ is the ideal, or universal, gas constant, equal to the product of the Boltzmann constant and the Avogadro constant, $R = 8.31\, J/molK^o$, and $T$ is the absolute temperature of the gas in $K^o$. The mass of one mole is $M = 0.028$ kg and $n = m/M$, where $m$ is the mass of the hot air inside the balloon. Thus the equation (5) can be rewritten in the following form:

$$pV = \frac{mRT}{M}. \tag{6}$$

Therefore, the mass of the hot air is

$$m = \frac{MpV}{RT}. \tag{7}$$

Since the inside air pressure $p$ equals outside atmospheric air pressure, the mass of the cold air (at the absolute temperature $T_0 = 288^o K$) equals

$$m_0 = \frac{MpV}{RT_0}. \tag{8}$$

Therefore, the payload of the balloon equals $mg - m_0 g$, where $g = 9.8\, m/s^2$. The total weight of the cabin and passengers is $400\, kg + 5 * 80\, kg = 800\, kg$. Assuming that the balloon is in equilibrium at the sea level,

$$mg - m_0 g = 800. \tag{9}$$

Your project consists of the following steps:

**Step 1.** Use equations (7)-(9) to find the temperature of the hot air that provides horizontal flight at the sea level.

**Step 2.** Use the linear regression and following table (https://www.avs.org/AVS/files/c7/c7edaedb-95b2-438f-adfb-36de54f87b9e.pdf) to find functions $T_0(h)$ and $p(h)$ that represent the dependence of the outside temperature $T_0$ and atmospheric pressure $p$ on the altitude $h$.

| Altitude Above the Sea Level in meters | Temperature in $K^o$ | Atmospheric Pressure in kPa |
|---|---|---|
| 0 | 288 | 101.33 |
| 153 | 287 | 99.49 |
| 305 | 286 | 97.63 |
| 458 | 285 | 95.91 |
| 610 | 284 | 94.19 |
| 763 | 283 | 92.46 |
| 0915 | 282 | 90.81 |
| 1068 | 281 | 89.15 |
| 1220 | 280 | 87.49 |
| 1373 | 279 | 85.91 |
| 1526 | 278 | 84.33 |
| 1831 | 276 | 81.22 |
| 2136 | 274 | 78.19 |
| 2441 | 272 | 75.22 |

**Step 3.** Find the temperature $T(h)$ that provides the horizontal flight of the balloon on the altitude $h$.
**Step 4.** Using the relationships between the altitude, outside temperature, atmospheric pressure, and the temperature of the hot air, on different levels of altitude find the rates of change of the altitude when the temperature of the hot air increases with the rate $2^o K$ per hour.

**Mini–project 2. Elasticity of Demand.** You are the sales manager of a big company. Your goal is to choose the right price of the product that maximizes the revenue. The revenue is the product of the demand and the price that are related by the price-demand equation

$$x = f(p).$$

Thus the revenue is

$$R(p) = pf(p). \tag{10}$$

Usually the increase of the price results in decrease of the demand. That is if you increase the price $p$ then the demand $f(p)$ decreases. However, it is unclear how it will affect the revenue. The question is how significantly the increase of the price will change the demand. If as a result of increase of the price the demand drops significantly then revenue will decrease, otherwise, if decrease of the demand is relatively small, the revenue will increase. The Elasticity of Demand answers this question.

Since we are interested in the change of the revenue, let us differentiate (10)

$$R'(p) = f(p) + pf'(p). \tag{11}$$

Therefore, the revenue increases if

$$f(p) + pf'(p) > 0, \tag{12}$$

that is

$$-\frac{pf'(p)}{f(p)} < 1. \tag{13}$$

The expression

$$E(p) := -\frac{pf'(p)}{f(p)} \tag{14}$$

is called the Elasticity of Demand. If $E(p) < 1$ the dependence of demand on price is relatively weak. In this case one says that the demand is inelastic. If $E(p) > 1$ then $R'(p) < 0$ and the demand is elastic. Finally, if $E(p) = 1$, then $R'(p) = 0$, and one says that the demand is unit.

In our introduction of the Elasticity of Demand we assumed that the price-demand equation defines an explicit function $x = f(p)$. However, in real applications this relationship may be implicit. In this mini–project you will discuss such cases.

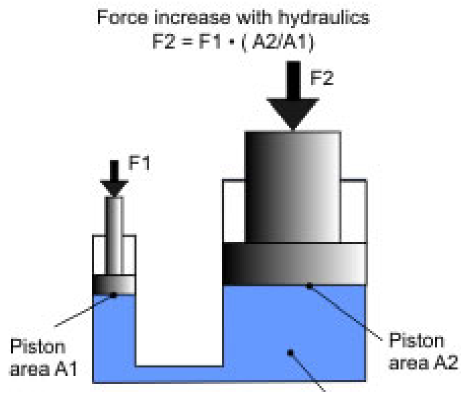
Plan of work:

i) choose several examples of implicit price-demand equations;

ii) for each example analyze how increasing the price affects the revenue;

iii) for each example introduce the Elasticity of Demand;

iv) try to develop the concept of the Elasticity of Demand in case of an implicit price-demand equation.




Press.jpg

Figure 1: Hydraulic Press

**Mini–project 3. Hydraulic Press.**

`http://hyperphysics.phy-astr.gsu.edu/hbase/pasc.html#hpcal`

In this project you will design a hydraulic press.

Wikipedia: "The hydraulic press depends on Pascal's principle: the pressure throughout a closed system is constant. One part of the system is a piston acting as a pump, with a modest mechanical force acting on a small cross-sectional area; the other part is a piston with a larger area which generates a correspondingly large mechanical force. Only small-diameter tubing (which more easily resists pressure) is needed if the pump is separated from the press cylinder.

Pascal's law: Pressure on a confined fluid is transmitted undiminished and acts with equal force on equal areas and at 90 degrees to the container wall.

A fluid, such as oil, is displaced when either piston is pushed inward. Since the fluid is incompressible, the volume that the small piston displaces is equal to the volume displaced by the large piston. This causes a difference in the length of displacement, which is proportional to the ratio of areas of the heads of the pistons, given that volume = area  length. Therefore, the small piston must be moved a large distance to get the large piston to move significantly. The distance the large piston will move is the distance that the small piston is moved divided by the ratio of the areas of the heads of the pistons. This is how energy, in the form of work in this case, is conserved and the Law of Conservation of Energy is satisfied. Work is force applied over a distance, and since the force is increased on the larger piston, the distance the force is applied over must be decreased."

Plan of work:

i) choose parameters of a hydraulic press such as the pistons areas, the pump force, and the designed speed of the lifting piston,

ii) find the fluid pressure under the piston of the pump,

iii) use Pascal's law to find the pressure under the lifting piston,

iv) find the lifting force of the hydraulic press,

v) use the Law of Conservation of Energy to find the relation between distances the pistons move,

vi) use this relation to find the relation between the speeds of pistons,

vii) find the speed of the lifting piston,

viii) calculate the power of the hydraulic press you designed,

ix) if your hydraulic press works from standard 120 v. outlet what is the electrical current through the wiring (you have to know it in order to choose correct fuses)?

**Mini–project 4. Wolf–Moose population.** The following diagram shows dynamics of wolf and moose population in Isle Royale, Mi. The goal of this project is to analyze relation between rates of change of wolf and moose population.

Your project may consist of the following steps:

**Step 1.** Use polynomial regression to express wolf population as a function of time $w(t)$.

**Step 2.** Use polynomial regression to express moose population as a function of time $m(t)$ .

**Step 3.** Find rates of change $w'(t)$ and $m'(t)$.

**Step 4.** Use related rates to estimate $\frac{dw}{dm}$.

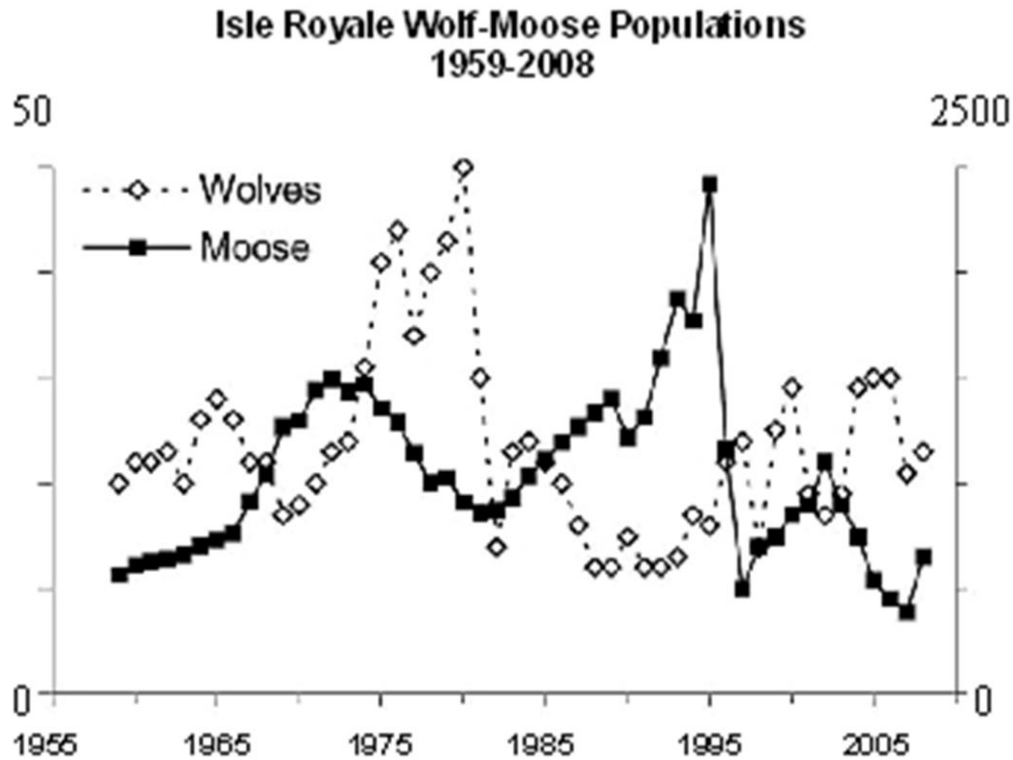





Figure 2: Wolf-Moose populations