**Kettering University Mathematics Olympiad For High School Students 2002, Sample Solutions**

1. (a) We want to write $29 - 12\sqrt{5} = x^2$. We predict that the form of $x$ is $(a - b\sqrt{5})$. Thus

$$29 - 12\sqrt{5} = a^2 - 2ab\sqrt{5} + 5b^2.$$

Hence $a^2 + 5b^2 = 29$ and $ab = 6$. Assuming that $a$ and $b$ are integers then the possible factors for 6 are

i. $a = \pm 1, b = \pm 6$
ii. $a = \pm 2, b = \pm 3$
iii. $a = \pm 3, b = \pm 2$
iv. $a = \pm 6, a = \pm 1$.

Upon trial and error, we see that $a = 3, b = -2$ or $a = -3, b = 2$. Hence

$$29 = 12\sqrt{5} = (3 - 2\sqrt{5})^2 = (-3 + 2\sqrt{5})^2.$$

**Remark:** One can also use the fact that $ab = 6 \Rightarrow b = \frac{6}{a}$ and substitute this expression into $a^2 + 5b^2 = 29$ to solve for $a$ and hence $b$.

(b) We want to write $10 - 6\sqrt{3} = x^3$. We predict that the form of $x$ is $(a - b\sqrt{3})$. Thus

$$10 - 6\sqrt{3} = a^3 - 3\sqrt{3}a^2b + 9ab^2 - 3\sqrt{3}b^3.$$

Hence $a^3 + 9ab^2 = 10$ and $a^2b + b^3 = 2$. One inspect that $a = 1, b = 1$ satisfy the two conditions. Hence

$$10 - 6\sqrt{3} = (1 - \sqrt{3})^3.$$

**Remark:** One can also use the fact that $b(a^2 + b^2) = 2 \Rightarrow b > 0$ and if we assume $a$ and $b$ are integers then $a^2b + b^3 = 2$ together with $b > 0$ implies $b = 1$. Consequently $a = 1$ and in fact it also satisfy $a^3 + 9ab^2 = 10$.

2. We are given

$$\begin{aligned} x + cy &= 1 \qquad (1) \\ cx + 9y &= 3 \qquad (2) \end{aligned}$$

Thus $c\times(1)$ - (2) gives

$$(c^2 - 9)y = c - 3 \quad \Longrightarrow \quad (c + 3)(c - 3)y = c - 3.$$

The above equation has no solution if $c = -3$ since in this case we have $0 = -6$ which is nonsense.

If $c = 3$ then Equation (1) and (2) becomes

$$\begin{aligned} x + 3y &= 1 \\ 3x + 9y &= 3 \quad \Longrightarrow \quad x + 3y = 1. \end{aligned}$$

Thus in this case, the two equations coincide and hence we have infinitely many solutions.

Thus the only value for which the given system has no solution is when $c = -3$.

**Remark:** One can also solve this problem geometrically. The two given equations are straight lines in the $xy$ plane. Hence it has no solution if the lines are parallel but do not coincide.

3. (a) We want to show there is a point $(x_1, y_1)$ which passes through all parabolas of the form $y = x^2 + 2ax + a$. Let $a$ and $b$ be two distinct real numbers. Then we want

$$\begin{aligned} y_1 &= x_1^2 + 2ax_1 + a, && (3) \\ y_2 &= x_1^2 + 2bx_1 + b. && (4) \end{aligned}$$

Thus Equation (1) - (2) gives

$$2ax_1 - 2bx_1 + (a - b) = 0 \quad \Longrightarrow \quad 2x_1(a - b) = -(a - b).$$

Since $a \neq b$ we must have $x_1 = -\frac{1}{2}$, so $y_1 = \frac{1}{4}$.

(b) To find the vertex of the parabola $y = x^2 + 2ax + a$, we complete the square giving $y = x^2 + 2ax + a^2 + a - a^2 = (x + a)^2 + (a - a^2)$. Thus the vertex is $(-a, a - a^2)$. So the equation of the parabola is $y = -x^2 - x$.

4. Denote the present ages of Miranda, Cathy, Stella, Eva, Lucinda and Dorothea by $M, C, S, E, L$ and $D$ respectively. Let $x$ denote the number of years between future and present ages. Then from Miranda's speech we gather that

$$\begin{aligned} M + C + S + E + L + D &= 5M, && (5) \\ S + x &= 3M, && (6) \\ M + x + D + x &= M + S + E + L + D, && (7) \\ E + x &= 3E, && (8) \\ L + x &= 2S + 1. && (9) \end{aligned}$$

From Equation (6) we see that $x = 3M - S$, using this we may rewrite the remaining equations as follow:

$$\begin{aligned} E + L + D &= 4M - C - S, && (10) \\ 7M - 2S + D &= M + S + E + L + D, && (11) \\ 2E &= (3M - S), && (12) \\ L &= 3S - 3M + 1. && (13) \end{aligned}$$

Substituting (10) into (11) and solving for $D$ gives

$$D = 2S - 2M - C. \tag{14}$$

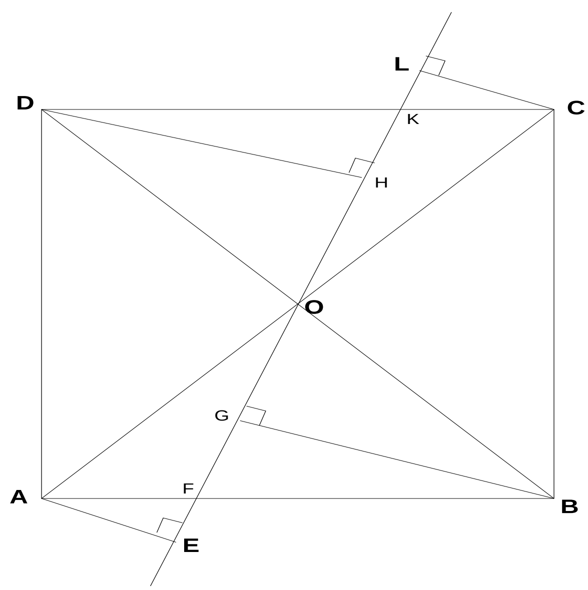
Multiply (5) by two and substitute Equations (12) through (14) to it gives (after simplification):

$$15M = 11S + 2.$$

Since we are given that Miranda's age is old then $15M$ must end in a $'5'$ which implies Stella's age must end in a $'3'$. Upon trial and error (and recalling that Miranda is in 10th grade) we found that Miranda's age is 17 and Stella's age is 23.

5. We wish to determine $(|AE|)^2 + (|BG|)^2 + (|CL|)^2 + (|DH)^2$. We note that this reduces to $2((|AE|)^2 + (|BG|)^2)$.

   Since $ABCD$ is a square this implies the sides of the square must be 10. Hence the diagonals must have length $\sqrt{(100)^2 + (100)^2} = \sqrt{20000}$. So $|AO| = 50\sqrt{2}$.




   We note that $\angle AOB = 90^0$. Thus $\angle GOB = 90 - \angle AOE$. Moreover, $\angle GOB + \angle OBG = 90^0$ hence $\angle OBG = \angle AOE$.

   Finally $AO = OB$ thus triangle $\triangle AEO \cong \triangle OGB$. So $OE = GB$, and we know that

$$|AE|^2 + |OE|^2 = |A0|^2 = 5000.$$

   Thus $|AE|^2 + |GB|^2 = 5000$ which implies the sum of squares of the distances from the cities to the highway is 10000.

6. (a) Denote the three coins by $A, B$ and $C$. (solid black hand arrows indicate weightings)

   - ☛ First Weighting: Coin $A$ against Coin $B$. There are two possible outcome.
     - ☞ Case 1: The pan is balance.
       This means both coins $A$ and $B$ are genuine. That is, we know $C$ is the counterfeit coin. We perform the following second weighting to determine whether coin $C$ is heavier or lighter than a genuine coin:
       ☛ Second Weighting: Weigh Coin $A$ against Coin $C$
     - ☞ Case 2: The pan is not balance. This means Coin $C$ is genuine and either coin $A$ or coin $B$ is a counterfeit. Without loss of generosity we can assume weight of Coin $A >$ weight of Coin $B$.
       We perform the following second weighting:
       ☛ Second Weighting: Weigh Coin $A$ against Coin $C$
       Again there are two possible outcome.
       - ☞ Case 1: The pan is balance.
         This means both coins $A$ and $C$ are genuine. That is, we know $B$ is the counterfeit coin. Moreover from the first weighting we know that Coin $B$ is lighter than a genuine coin.
       - ☞ Case 2: The pan is not balance.
         Thus Coin $A$ is the counterfeit and it is heavier than a genuine coin.

   Hence we have found the counterfeit coin (and whether it is lighter or heavier) in 2 weightings.

   (b) Denote the 12 coins by $A, B, C, D, E, F, G, H, I, J, K$ and $L$ respectively. (solid black hand arrow indicate weightings)
   We begin by dividing the 12 coins into 3 groups of 4 coins.
   Group 1: Coin $A, B, C, D$
   Group 2: Coin $E, F, G, H$
   Group 3: Coin $I, J, K, L$

   - ☛ First Weighting: Group 1 coins against Group 2 coins. There are two possible outcome.
     - ☞ Case 1: The pan is balance.
       This means Coins $A, B, C, D, E, F, G, H$ are all genuine. Thus the counterfeit coin must be in Group 3.
       ☛ Second Weighting: Weigh Coins $A, I$ against Coins $J, K$.
       There are three possibilities:
       - ☞ If weight of Coins $A, I =$ weight of Coins $J, K$.
         This means Coin $L$ is the counterfeit coin and we can determine whether it is heavier or lighter than a genuine coin

by doing:

☚ Weigh Coin $A$ against Coin $L$.

☞ If weight of Coins $A, I >$ weight of Coins $J, K$.

This means counterfeit coin is either Coin $I, J$ or $K$.

☚ Third Weighting: Weigh Coin $J$ against Coin $K$

☞ If the pan is balance, this means Coin $I$ is the counterfeit and from second weighting we can conclude Coin $I$ is heavier than a genuine coin.

☞ If the pan is not balance then Coin $I$ is genuine and from second weighting we can conclude counterfeit coin is lighter thus from the third weighting we can conclude which is the counterfeit coin.

☞If weight of Coins $A, I <$ weight of Coins $J, K$.

This means again counterfeit coin is either Coin $I, J$ or $K$. We can follow the same procedure as above to determine the counterfeit coin and whether it is heavier or lighter than a genuine coin.

☞ Case 2: The pan is not balance. This means the counterfeit coins must be in Group $1(A, B, C, D)$ or $2(E, F, G, H)$ and Group 3 $(I, J, K, L)$ consist of all genuine coins. Without lost of generosity we can assume Weight of Group 1 coins $>$ Weight of Group 2 coins.

☚ Second Weighting: Weigh Coins $A, B, E$ against Coins $D, F, J$. There are three possibilities:

☞ weight of coins $A, B, E =$ weight of coins $D, F, J$.

This means the counterfeit coin is either coin $C, G$ or $H$.

☚ Third Weighting: Weigh Coin $G$ against Coin $H$

☞ If the pan is balance, this means Coin $C$ is the counterfeit and from first weighting we can conclude Coin $C$ is heavier than a genuine coin.

☞ If the pan is not balance then Coin $C$ is genuine and from first weighting we can conclude counterfeit coin is lighter thus from the third weighting we can conclude which is the counterfeit coin.

☞ weight of coins $A, B, E >$ weight of coins $D, F, J$.

This means the counterfeit coin is either coin $A, B$ or $F$.

☚ Third Weighting: Weigh Coin $A$ against Coin $B$

☞ If the pan is balance, this means Coin $F$ is the counterfeit coin and from first (or third) weighting we can conclude that Coin $F$ is lighter than a genuine coin.

☞ If the pan is not balance, then Coin $F$ is genuine and from the first weighting we can conclude that the counterfeit coin is heavier thus from the third weighting we can conclude which is the counterfeit coin.

☞ weight of coins $A, B, E <$ weight of coins $D, F, J$.
This means the counterfeit coin is either coin $D$ or coin $E$.

☛ Third Weighting: Weigh Coin $I$ against Coin $D$.
☞ If the pan is balance, this means Coin $E$ is the counterfeit coin and from first (or third) weighting we can conclude the coin is lighter than a genuine coin.
☞ If the pan is not balance, then since Coin $I$ is genuine we can conclude whether the counterfeit coin $D$ is lighter or heavier than a genuine coin.

In all cases, we are able to find the counterfeit coin (and whether it is lighter or heavier) in 3 weightings.