Kettering University Mathematics Olympiad For High School Students 2003, Sample Solutions

1. Solving for y in the first equation gives $y = 3 - x$. Substituting into second equation gives

$$\begin{aligned} & 4x(3-x) - z^2 = 9 \\ \Longrightarrow\quad & z^2 + 4x^2 - 12x + 9 = 0 \\ \Longrightarrow\quad & z^2 + (2x-3)^2 = 0. \end{aligned}$$

Hence the only solution for the above equation is $x = \frac{3}{2}$ and $z = 0$. The corresponding y value is also $\frac{3}{2}$.

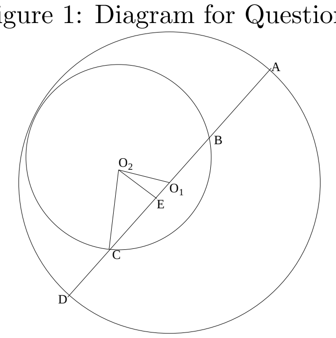
2. Let P_0 denote the amount of money Mr. Money has in his bank account. Let P_i denote the amount of money remaining in Mr. Money's account after i years. Then we know that

$$\begin{aligned} P_1 &= 3/4P_0 \\ P_2 &= \frac{12}{10}P_1 = \frac{9}{10}P_0 \\ P_3 &= \frac{9}{10}P_2 = \frac{81}{100}P_0 \\ P_4 &= \frac{12}{10}P_3 = \frac{243}{250}P_0. \end{aligned}$$

Hence Mr. Money's account has decreased by $\frac{7}{250}P_0$. That is, it decreases by 2.8%.

3. A diagram corresponding to this problem is shown in Figure 1. Here O_1 denotes the center of the larger circle and O_2 denotes the center of the smaller circle. We are given that $|AB| : |BC| : |CD| = 3 : 7 : 2$.

Figure 1: Diagram for Question 3



We may assume that $|AB| = 3$, $|BC| = 7$ and $|CD| = 2$. Let R, r denotes the radius of the larger and smaller circle respectively. Then $2R = |AB| + |BC| + |CD| = 12$, hence $R = 6$.

Let $|O_2E|$ be the perpendicular to the line AD then $|EC| = |EB|$. Furthermore, by Pythagoras Theorem we have

$$\begin{aligned} |O_2E|^2 + |EC|^2 &= |O_2C|^2 \\ |O_2E|^2 + |EO_1|^2 &= |O_2O_1|^2. \end{aligned}$$

Hence

$$|O_2C|^2 - |EC|^2 = |O_2O_1|^2 - |EO_1|^2. \tag{1}$$

We know that

$$\begin{aligned} |O_2C| &= r, \\ |O_2O_1| &= R - r = 6 - r, \\ |EC| = \frac{1}{2}|BC| = \frac{7}{2}, \\ |EO_1| &= R - |CD| - |EC| = 6 - 2 - \frac{7}{2} = \frac{1}{2}. \end{aligned}$$

Substituting the above expressions into Equation 1 gives

$$r^2 - \frac{49}{4} = 36 - 12r + r^2 - \frac{1}{4} \quad \Rightarrow \quad 12r = 48 \quad \Rightarrow \quad r = 4.$$

Hence the ratio $\frac{r}{R} = \frac{2}{3}$.

4. The given equation implies that $xy = 19(x+y)$. To have integer solutions at least one of x or y must be divisible by 19. Since the equation is symmetric we can assume $x = 19k$ where k is an integer. Then

$$xy = 19(x+y) \quad \Rightarrow \quad ky = 19k + y.$$

From the last equality we see that $19k + y$ must be divisible by k. Since clearly $19k$ is divisible thus $y = km$ where m is an integer. Hence

$$ky = 19k + y \quad \Rightarrow \quad km = 19 + m \quad \Rightarrow \quad m(k-1) = 19.$$

Since m and k are both integers there are only 4 possible combinations for which the above equation holds:

$$\begin{aligned} &m = 19, k-1 = 1 \quad \Rightarrow \quad x = 38, y = 38 \\ &m = -19, k-1 = -1 \quad \Rightarrow \quad x = 0, y = 0 \text{ reject} \\ &m = 1, k-1 = 19 \quad \Rightarrow \quad x = 380, y = 20 \\ &m = -1, k-1 = -19 \quad \Rightarrow \quad x = -342, y = 18. \end{aligned}$$

Hence we have 5 possible solutions (x, y). Namely,
$(38, 38)$, $(380, 20)$, $(20, 380)$, $(-342, 18)$, and $(18, -342)$.

5. Let $S_1 = \{1, 2, 3, 10, 11, 12\}$ and $S_2 = \{4, 5, 6, 7, 8, 9\}$. First note that the numbers in S_1 cannot be placed adjacent to each other. Hence there must exists exactly one number from the set S_2 between any two numbers in S_1. But the number 4 can only be placed next to the number 1. Thus it is impossible to construct the arrangement stated.

6. Suppose by way of contradiction that the overlapping area between any 2 rectangles is less than $\frac{1}{9}$ square mile. Let A denote the total non-overlapping area covered by the 9 small rectangles. So

$$A > 1 + \frac{8}{9} + \frac{7}{9} + \frac{6}{9} + \frac{5}{9} + \frac{4}{9} + \frac{2}{9} + \frac{2}{9} + \frac{1}{9} = 5.$$

This contradicts the fact that the 9 small rectangles is totally enclosed in the larger rectangles. Hence our assumption is incorrect. That is, there must exists two rectangles whose overlapping area is greater than or equal to $\frac{1}{9}$ square miles.