Kettering University Mathematics Olympiad For High School Students 2005

1. Solve the equation:
$$(1+x^2)(1+x^4) = 4x^3.$$

2. Nick and John play the following game. They put 100 pebbles on the table. During any move a player takes at least one and not more than eight pebbles. Nick makes the first move, then John makes his move, then Nick makes a move again and so on. The player who takes the last pebble is the winner of the game. Can you offer Nick some strategy to win the game, can you offer John such a strategy?

3. Prove that:
$$\sin(x) + \sin(3x) + \sin(5x) + \cdots + \sin(11x) = \frac{1-\cos(12x)}{2\sin(x)}.$$

4. Nick played the following game. He took 7 pieces of paper and cut some of them in 7 pieces each. Then he mixed all the pieces together, took some of them and cut again in 7 pieces each, and so on. After finishing this game he counted the number of all the pieces of paper (of different size) and told his older brother John that the number of pieces 2000. After thinking a while John told Nick that the number of pieces could not be 2000. Why did John decide that there was an error?

5. Find the angles of the triangle ABC in which the height, the median and the bisector taken from the vertex A, divide the angle A into four equal angles.

6. One hundred cities are connected by airlines (every airline connects two cities, some cities are connected and some are not).

   (a) What minimal number of airlines is required to create a scheme connecting any two cities with not more that k changes of plane $(0 < k < n)$? Give an example of such a connection scheme.

   (b) Prove that if the number of airlines is 4852, then for any connection scheme, it is possible to travel from any one of these cities to another one using these airlines and changing planes as many times as needed.