Kettering University Mathematics Olympiad For High School Students 2006, Sample Solutions

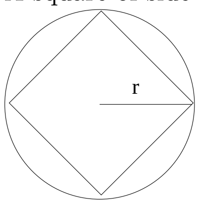
1. (*Solution by Nicholas Triantafillou, a 4th-7th finisher*)
Since we know that the product of the three boys ages is 36, we make a list of the possible ages of the three boys, from youngest to oldest. Thus,

Table 1: Table of Possible Ages and the Sum of the Ages

Ages	Sum
1,1,36	38
1,2,18	21
1,3,12	16
1,4,9	14
1,6,6	13
2,2, 9	13
2,3,6	11
3,3,4	10

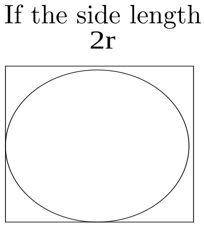
one of these sums must be the number of windows in the chemist and mathematician's high school. However, since the mathematician is left with indecision about the children's ages, the sum must be repeated. The only value to satisfy this is 13. Therefore the children are either 1,6 and 6 or 2,2 and 9. Since we know that one child is oldest, however, the combination 1,6,6 is not possible. The three son's ages are 2,2 and 9.

2. (*Solution by Philip Hu, Philip placed second in the 2006 Kettering Math Olympiad.*)
A square of side length $\sqrt{2}r$ can be inscribed in the incircle.

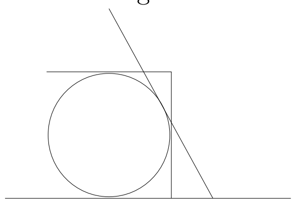


If the square is oriented so that one of its sides is parallel to the base of the triangle (consider the side that is closer to the base since two sides with ultimately be parallel), that side will not touch the base since it subtends a 90^o arc of the incircle. This square is obviously too small.

If the side length is $2r$, it can be circumscribed around the incircle:



If it is oriented so that one side is parallel to the base, (again, consider the side that is closer to the base since two sides with ultimately be parallel), the two vertices on that side will be on the base since both the side and the base are tangent to the incircle. Since two angle of a triangle can't be greater than or equal to 90^o, consider one side of the triangle that intersects the base such that it forms an acute angle in the triangle.



Since it is tangent to the incircle and not parallel to either sides parallel to the base or sides perpendicular to the base, it must cut off a top corner of the square. Thus, the square is too large to be inscribed in the triangle as desired.

3. (*Sample Solution:*)
Denote the 100 numbers by: $x_1, x_2, \ldots, x_{100}$. Consider the following sums:

$$\begin{aligned} S_1 &= x_1 \\ S_2 &= x_1 + x_2 \\ \vdots & \quad \vdots \\ S_k &= x_1 + x_2 + \cdots x_k \\ \vdots & \quad \vdots \\ S_{100} &= x_1 + x_2 + \cdots x_{100}. \end{aligned}$$

We can represent $S_i = 100k_i + r_i$ where r_i denotes the remainder of S_i when divided by 100, hence $0 \leq r_i \leq 99$. Since we have exactly 100 sums, there is either some i such that $r_i = 0$ or there exists at least 2 indices i and j, $i > j$ such that $r_i = r_j$. If there is i such that $r_i = 0$ then S_i is divisible by 100 and we are done. For the remaining possibility, set $S = S_i - S_j = x_{j+1} + x_{j+2} + \cdots x_i = 100k_i - 100k_j$, then clearly S is divisible by 100.

4. (*Solution by Saurabh Pandey, a 4th-7th finisher*)

- If both a and b are odd. Then a^b will be odd and b^a will be odd, so $a^b + b^a$ will be even. Contradiction.
- If a and b are both even, then a^b will be even and b^a will be even, so $a^b + b^a$ will be even. Contradiction.
- So one of a or b must be odd; the other is even. WLOG, let a be odd and b be even. The only even prime number is 2 so b is 2.

Our problem is now reduced to finding, for which primes a is $a^2 + 2^a$ a prime.
If $a = 3$, we have $3^2 + 2^3 = 17$, a prime number.
If a is a prime greater than 3, we have

$$a \equiv 1(\text{mod } 3) \text{ or } a \equiv 2(\text{mod } 3) \tag{1}$$

because if $a \equiv 0(\text{mod } 3)$, and $a > 4$, then a would be a multiple of 3 and wouldn't be prime anymore.

We then see that $a^2 \equiv 1(\text{mod } 3)$ by squaring both congruences in Condition (1) and taking mod 3.

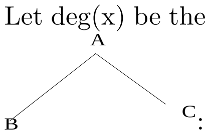
2^a when a is a prime greater than 3, can be rewritten as 2^{2c+1} where c is integral and greater than 0. Now

$$2^{2c+1} = 2(2^{2c}) = 2(4^c) = 2(3+1)^c \equiv 2(1)^c(\text{mod } 3) \equiv 2(\text{mod } 3)$$

So $2^a + a^2 \equiv 2 + 1(\text{mod } 3) \equiv 0(\text{mod } 3)$ for all primes a greater than 3. This implies $2^a + a^2$ is a multiple of 3 and thus not prime. Contradiction. So $a = 3$ and one solution is $(a, b) = (3, 2)$. The other solution is just a permutation of the first solution $(a, b) = (2, 3)$.

5. (*Solution by Chaitanya Malla, Chaitanya placed third in the 2006 Kettering Math Olympiad.*)
We can convert the problem to Graph Theory terms. Let each airport be a vertex and draw an edge between two vertices (airports) if the yare connected. We are given that the graph is connected and have to prove that removing a vertex does not disconnect the graph.

Let deg(x) be the degree (# of connections at x) of vertex x. For example



: deg(A)=2, deg(B)=deg(C)=1.

We consider 2 cases: (i) There is a cycle in the graph. It there is a cycle $a_1, a_2, \ldots, a_i, a_i$ then removing vertex a_j only removes the edges $a_{j-1}a_j$ and $a_j a_{j+1}$ and the remaining vertices in the cycle are still connected.

(ii) If there is no cycle, then the graph is called a tree. We need to show that there exists a vertex x such tath deg(x)=1 since removing x will not disconnect the graph. This is clearly true for N=2 vertices. Assume for some k that the tree with k vertices has a vertex of degree 1. Now consider adding an extra vertex A to the tree with k vertices. If deg(A)=1, then we are done. Assume to the contrary that deg(A)$\geq$ 2. Then A is joined to two distinct vertices B and C. But since B and C are part of the original connected graph, there exists a path from B to C not through A. Therefore, we have a cycle which is a contradiction since we said there were no cycles. Therefore, deg(A)=1 and by induction we have proven that a tress always has a vertex of degree 1.

6. (*Solution by Daniel Echlin, Daniel is the winner of the 2006 Kettering Math Olympiad.*)

We show $x^n + y^n = z^n$, $z \leq n$ is insoluble in $\mathbb{Z}$ by showing there is a problem with the size of $x^n + y^n$. We need to show $x^n + y^n < m^n$. Obviously $x, y < m$ or else we get $x^n < 0$ or $y^n < 0$ but $x, y > 0$. The biggest choice we can make is $x = y = m - 1$ or $2(m-1)^n < m^n$. Now

$$\begin{aligned} & 2(m-1)^n < m^n \\ \Leftrightarrow\ & \sqrt[n]{2}(m-1) < m \\ \Leftrightarrow\ & \sqrt[n]{2}m - \sqrt[n]{2} < m \\ \Leftrightarrow\ & (\sqrt[n]{2} - 1)m < \sqrt[n]{2} \\ \Leftrightarrow\ & m < \frac{\sqrt[n]{2}}{\sqrt[n]{2} - 1} \end{aligned}$$

Because $m \leq n$, if we can show $n < \dfrac{\sqrt[n]{2}}{\sqrt[n]{2} - 1}$, we're in good shape.

$$\begin{aligned} & n < \frac{\sqrt[n]{2}}{\sqrt[n]{2} - 1} \\ \Leftrightarrow\ & \frac{1}{n} > \frac{\sqrt[n]{2} - 1}{\sqrt[n]{2}} \qquad \text{(because both sides are positive and } f(x) = \frac{1}{x} \text{ is decrasing on } (0, \infty), \\ & \qquad \text{we can do this and reverse the inequality)} \\ \Leftrightarrow\ & \frac{1}{n} > 1 - \frac{1}{\sqrt[n]{2}} \\ \Leftrightarrow\ & \frac{1}{n} - 1 > -\frac{1}{\sqrt[n]{2}} \\ \Leftrightarrow\ & 1 - \frac{1}{n} < \frac{1}{\sqrt[n]{2}} \\ \Leftrightarrow\ & \frac{n-1}{n} < \frac{1}{\sqrt[n]{2}} \\ \Leftrightarrow\ & \left(\frac{n-1}{n}\right)^n < \frac{1}{2} \end{aligned}$$

We may now apply the binomial theorem:

$$\begin{aligned} \left(\frac{n-1}{n}\right)^n &= \frac{n^n - nn^{n-1} + \binom{n}{2} n^{n-2} - \cdots \pm (-1)^n}{n^n} \\ &= \frac{n^n - n^n + \cdots \pm (-1)^n}{n^n} \\ &= \binom{n}{2} n^{-2} - \binom{n}{3} n^{-3} + \cdots \pm (-1)^n n^{-n} \end{aligned}$$

If $n \geq 2$, this expression is obviously bounded by $\frac{\binom{n}{2}}{n^2}$ which is less than $\frac{1}{2}$. Hence whenever $x, y < m \leq n$, $x^n + y^n < m^n$.