Kettering University Mathematics Olympiad For High School Students 2008

1. (*Solution by Kevin Wang, a 4th-7th finisher*)
We can consider both terms of $\max\{a, b\}$ and $\max\{c, d\}$ and compare to the value of $\max\{a + c, b + d\}$.
Case 1: $a \geq b, c \geq d$
Then $\max\{a, b\} = a$, $\max\{c, d\} = c$. Since $a + c \geq b + d$, $\max\{a + c, b + d\} = a + c$. So $a + c = a + c$ and $\max\{a + c, b + d\} \leq \max\{a, b\} + \max\{c, d\}$.

Case 2: $a < b, c < d$
Then $\max\{a, b\} = b$, $\max\{c, d\} = d$. Since $b+d > a+c$, $\max\{a+c, b+d\} = b+d$. So $b+d = b+d$ and $\max\{a+c, b+d\} \leq \max\{a, b\}+\max\{c, d\}$.

Case 3: $a \geq b, c \leq d$
Then $\max\{a, b\} = a$, $\max\{c, d\} = d$. Since $c \leq d$, $a + c \leq a + d$, Since $b \leq a, b + d \leq a + d$. Thus $\max\{a + c, b + d\} \leq \max\{a, b\} + \max\{c, d\}$ as both $a + c$ and $b + d$ are $\leq a + d$.

Case 4: $a \leq b, c \geq d$
Then $\max\{a, b\} = b$, $\max\{c, d\} = c$. Since $a \leq b$, $a + c \leq b + c$, Since $d \leq c, b + d \leq b + c$. Thus $\max\{a + c, b + d\} \leq \max\{a, b\} + \max\{c, d\}$ as both $a + c$ and $b + d$ are $\leq b + c$.

2. (*Solution by Siddhant Dogra, a 4th-7th finisher*)
If a number is divisible by 3333333, it is divisible by 1111111 and 3. Let the answer be m, so $m = 1111111n$ where $3|n$ and n is greater than 0 and is an integer.

For m's digits to all be ones, the units digit of n must be 1. Now, when we multiply by 1111111 by n, we must make sure there is no carrying in the seven right most numbers, otherwise m will not have only ones as digits. Therefore, we must place size zeroes to the left of the units digit in n. Now, to keep all the digits as ones, we put a 1 after the leftmost zero. So the last eight digits of n are ...10000001. It is well know that the sum of digits of some number must be divisible by 3 for the number to be divisible by 3. Thus, we must have another 1.

Applying the same reasoning again, the last fifteen digits of n must be ...100000010000001. However, now $3|n$ and we may stop. Thus $n = 100000010000001$ gives the smallest m which is 111111111111111111111 where there are 21 ones.

3. (*Solution by David Lu, a 4th-7th finisher*)
$\sqrt{x} = \sqrt{2560} - \sqrt{y}$. So $x = 2560 + y - 32\sqrt{10y}$. This must be an integer, therefore $32\sqrt{10y}$ must be an integer and y must be ten times a square or $y = 10b^2$.

The same reasoning with x and y switched shows that $x = 10a^2$ (x is also ten times a square.) Assume a, b are non-negative. Then we have $a\sqrt{10} + b\sqrt{10} = 16\sqrt{10} \Rightarrow a + b = 16$. And we can see that there are 16 solutions:

$$\begin{aligned} & (a,b) = (0,16),(1,15),(2,14),(3,13),\ldots,(15,1),(16,0) \\ \Rightarrow\ & (x,y) = (0,2560),(10,2250),(40,1960),\ldots,(2250,10),(2560,0) \end{aligned}$$

4. (*Solution by Neil Gurram, Neil placed third in the 2009 Kettering Math Olympiad*)
Let us have the following definition.

Define $A_n = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots \frac{1}{2}\sqrt{\frac{1}{2}}}}}$, where there are n square roots in the expression above. So, we have $A_1 = \sqrt{\frac{1}{2}}$, $A_2 = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}$ and so forth. But the important fact we need to utilize is that $A_n = \sqrt{\frac{1}{2} + \frac{1}{2}A_{n-1}}$. Now, notice that $A_1 = \sqrt{\frac{1}{2}} = \cos 45^o$. Set $\theta_1 = 45^o$ then $A_1 = \cos\theta_1$. Observe then that $A_2 = \sqrt{\frac{1+\cos\theta_1}{2}} = \cos\left(\frac{\theta_1}{2}\right)$ by the half-angle formula for cosine. So $A_2 = \cos 22.5^o$. Let $A_2 = \cos\theta_2$ where $\theta_2 = 22.5^o$. Now, we prove by induction that $A_n = \cos\left(\frac{45^o}{2^{n-1}}\right)$.

We showed already that $A_1 = \cos 45^o = \cos\left(\frac{45^o}{2^0}\right)$. Now assume $A_n = \cos\left(\frac{45^o}{2^{n-1}}\right)$ for $n = k$. To prove for $n = k+1$.

$$A_{k+1} = \sqrt{\frac{1 + A_k}{2}} = \sqrt{\frac{1 + \cos\left(\frac{45^o}{2^{k-1}}\right)}{2}} = \cos\left(\frac{45^o}{2^k}\right) = \cos\left(\frac{45^o}{2^{(k+1)-1}}\right)$$

So, we have proven by induction that $A_n = \cos\left(\frac{45^o}{2^{n-1}}\right)$.

5. (*Solution by Alex Song, Alex is co-winner in the 2009 Kettering Math Olympiad*)
First of all, note the for fixed A, B, for XAB to be isosceles, X must be on the perpendicular bisector of AB or on a circle with center A or B and radius AB. Now, we split into two cases. (Let X be a new airport.)

   Case 1: N is even
   Consider the points in a long diagonal, A and B. For XAB to be isosceles and X in the polygon, X must be on the perpendicular bisector of AB. (as both possible circles are outside of the polygon). Similarly, it must be on the perpendicular bisector of all long diagonals and hence it is the center. This is a contradiction to (i) as then XAB will be collinear. Thus, if N is even, 0 airports can be added.

   Case 2: N is odd.
   Take any point C. Now, let the long diagonals be CD and CE. ($CD = CE$) To have XCD be isosceles, once again X is on the perpendicular bisector of CD. Thus $XC = XD$. Similarly, $XC = XE$, so X is the circumcentre of triangle CDE and thus the center. Because the center satisfies all conditions, it could be the only possible airport.

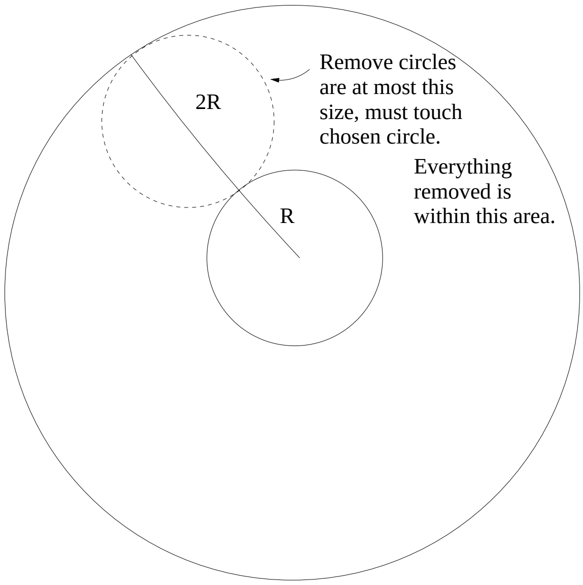
   Thus, for even N, 0 airports can be added and for odd N, 1 airport can be added.

6. (*Solution by Allen Yuan, Allen is co-winner in the 2009 Kettering Math Olympiad*)
Yes it is possible!

   Consider the following algorithm for choosing circles.
   Step 1: Choose a circle of maximal radius.
   Step 2: circle and all circles touching it from the picture. *Step 3:* Repeat step 1 if a circle remains - otherwise end algorithm. (The algorithm terminates due to finiteness.)

   I claim this algorithm chooses a disjoint subset of our original family of circles that has area of at least 1 m^2.

   Denote the circles $c_1, c_2, \ldots, c_k$ with radii $r_1, r_2, \ldots, r_k$ respectively. So that they are in order of choosing (implying $r_1 \geq r_2 \geq \ldots \geq r_k$)



Suppose c_i and c_j intersect, $i < j$. Then, in step 2 of choosing c_i we would have removed c_j, a contradiction. Thus, c_1 through c_k are disjoint. Now let us start running the algorithm.

Let U be a variable for the area of the union of remaining circles. Let A be a variable for the total area of the chosen circles so far. When we choose a circle of radius R, since we always choose the maximal circle, the union of the area of the circles removed in step 2 is at most $(3R)^2\pi = 9\pi R^2$. Thus, the variable U decreases at most $9\pi R^2$. On the other hand, variable A increases by exactly πR^2. Since initially $U \geq 9m^2$ and $A = 0$ and at the end $U = 0$ (the algorithm terminates), we have

$$9m^2 \leq U \leq 9\pi(r_1^2 + r_2^2 + \cdots + r_k^2).$$

Thus

$$A = (r_1^2 + r_2^2 + \cdots + r_k^2)\pi \geq \frac{9m^2}{9} = 1m^2$$

as desired.