Solutions.

Problem 1. Solve the equation:

$$\frac{\sqrt{x^2-2x+1}}{x^2-1}+\frac{x^2-1}{\sqrt{x^2-2x+1}}=\frac{5}{2}.$$

Solution of Problem 1.

First we observe that $x \neq \pm 1$ and

$\sqrt{x^2-2x+1}=\sqrt{(x-1)^2}=|x-1|$.

Denote: $t=\frac{x^2-1}{|x-1|}$.

Then,

$$t+\frac{1}{t}=\frac{5}{2}.$$

Therefore,

$$t^2-\frac{5}{2}t+1=0,$$

and

$$\left[\begin{array}{l} t=2 \\ t=\frac{1}{2} \end{array}\right. .$$

1. $t=2$.

$$\frac{x^2-1}{|x-1|}=2$$

i) If $x>1$, then $|x-1|=x-1$, and

$$\frac{x^2-1}{|x-1|}=\frac{x^2-1}{x-1}=x+1=2.$$

Thus $x=1$ but it cannot be a solution because $x \neq \pm 1$.

i) If $x<1$, then $|x-1|=-(x-1)$, and

$$\frac{x^2-1}{|x-1|}=-\frac{x^2-1}{x-1}=-(x+1)=2.$$

Thus $x=-3$ is a solution.

2. $t=\frac{1}{2}$.

$$\frac{x^2-1}{|x-1|}=\frac{1}{2}$$

i) If $x>1$, then $|x-1|=x-1$, and

$$\frac{x^2-1}{|x-1|}=\frac{x^2-1}{x-1}=x+1=\frac{1}{2}.$$

Thus $x=-\frac{1}{2}$ but it cannot be a solution since $x>1$.

i) If $x<1$, then $|x-1|=-(x-1)$, and

$$\frac{x^2-1}{|x-1|}=-\frac{x^2-1}{x-1}=-(x+1)=\frac{1}{2}.$$

Thus $x = -\frac{3}{2}$ is a solution.

Therefore, there are two solutions $x = -3$ and $x = -\frac{3}{2}$.

Problem 2. Solve the inequality:

$$\frac{1 - 2\sqrt{1 - x^2}}{x} \leq 1.$$

Solution of Problem 2.

First we observe that $x \neq 0$ and $-1 \leq x \leq 1$. So there are two cases to be considered: $-1 \leq x < 0$ and $0 < x \leq 1$.

1) $0 < x \leq 1$.

Multiplying both sides of the inequality by x one gets:

$$1 - 2\sqrt{1 - x^2} \leq x.$$

Then,

$$1 - x \leq 2\sqrt{1 - x^2}.$$

Since $1 - x \geq 0$, the previous inequality is equivalent to

$$(1 - x)^2 \leq 4(1 - x^2)$$

and to

$$1 - x \leq 4(1 + x).$$

Thus in addition to $0 < x \leq 1$ one has $x \geq -\frac{3}{5}$. Therefore the solution in the first case is the interval $(0, 1]$.

2) $-1 \leq x < 0$.

Multiplying both sides of the inequality by x one gets:

$$1 - 2\sqrt{1 - x^2} \geq x.$$

Then,

$$1 - x \geq 2\sqrt{1 - x^2}.$$

Since $1 - x \geq 0$, the previous inequality is equivalent to

$$(1 - x)^2 \geq 4(1 - x^2)$$

and to

$$1 - x \geq 4(1 + x).$$

Thus in addition to $-1 \leq x < 0$ one has $x \leq -\frac{3}{5}$. Therefore the solution in the second case is the interval $[-1, -\frac{3}{5}]$.

So the solution to the inequality is $[-1, -\frac{3}{5}] \cup (0, 1]$.

Problem 3. Let $ABCD$ be a convex quadrilateral such that the length of the segment connecting midpoints of the two opposite sides AB and CD equals $\frac{|AD|+|BC|}{2}$. Prove that AD is parallel to BC.

Solution of Problem 3.

Denote by E and F the midpoints of AB and CD. Then $|AE| = |EB|$ and $|CF| = |FD|$.

Connect vertices A and C by a segment AC. Denote by G the midpoint of AC. Since $|AE| = |EB|$ and $|AG| = |GC|$ one has $EG||BC$ and $|EG| = \frac{1}{2}|BC|$. Similarly, since $|DF| = |FC|$ and $|AG| = |GC|$ one has $GF||AD$ and $|GF| = \frac{1}{2}|AD|$.

Thus,

$$|EG| + |GF| = \frac{|BC| + |AD|}{2} = |EF|.$$

By the triangle inequality it implies that E, G, and F belong to the same line. Therefore, $EF||BC$ and $EF||AD$, which implies that $BC||AD$.

Problem 4. Solve the equation:

$$\frac{1}{\cos x} + \frac{1}{\sin x} = 2\sqrt{2}.$$

Solution of Problem 4.

First we observe that $x \neq \frac{\pi n}{2}$, where n ia any integer. Since

$$\sin x + \cos x = 2\sqrt{2} \sin x \cos x$$

one gets

$$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = 2 \sin x \cos x.$$

This implies

$$\sin\left(x + \frac{\pi}{4}\right) - \sin 2x = 0.$$

Using the identity

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

one gets

$$2\sin\left(\frac{x-\frac{\pi}{4}}{2}\right)\cos\left(\frac{3x+\frac{\pi}{4}}{2}\right) = 0.$$

Therefore,

$$\left[\begin{array}{l} \frac{x-\frac{\pi}{4}}{2} = \pi n \\ \frac{3x+\frac{\pi}{4}}{2} = \frac{\pi}{2} + \pi n \end{array}\right. .$$

Thus, the solutions are

$$\left[\begin{array}{l} x = \frac{\pi}{4} + 2\pi n \\ x = \frac{\pi}{4} + \frac{2\pi n}{3} \end{array}\right. .$$

So the final answer is

$$x = \frac{\pi}{4} + \frac{2\pi n}{3},$$

where n is any integer.

Problem 5. Long, long ago, far, far away there existed the Old Republic Galaxy with a large number of stars. It was known that for any four stars in the galaxy there existed a point in space such that the distance from that point to any of these four stars was less than or equal to R. Master Yoda asked Luke Skywalker the following question: Must there exist a point P in the galaxy such that all stars in the galaxy are within a distance R of the point P? Give a justified argument that will help Luke answer Master Yoda's question.

Solution of Problem 5.

Let us consider the smallest ball that contains all stars. Denote the radius of this ball by r. To find such a ball one can start with a large ball that contains all stars inside it and then shrink it changing the position of its center if needed until one gets the ball with the minimal radius. To be the ball of the minimal radius that cannot be made smaller this ball should satisfy either one of the following three conditions:

1) there are two stars that are the endpoints of one of diameters of the ball,

2) there are three stars that are vertices of an acute triangle inscribed in a great circle of the ball (a great circle is the intersection of the surface of the ball with the plane passing throw the center of the ball),

3) there are four stars that are vertices of a tetrahedron inscribed in the ball, the center of which is inside the tetrahedron.

By the assumption of the problem these two (case 1), or three (case 2), or four (case 3) stars are inside of some ball of the radius R. Therefore, it is necessary that $r \leq R$. So all stars are inside the ball of the radius $r \leq R$ and the point P is the center of that ball.

Problem 6. The Old Republic contained an odd number of inhabited planets. Some pairs of planets were connected to each other by space flights of the Trade Federation, and some pairs of planets were not connected. Every inhabited planet had at least one connection to some other inhabited planet. Luke knew that if two planets had a common connection (they are connected to the same planet), then they have a different number of total connections. Master Yoda asked Luke if there must exist a planet that has exactly two connections. Give a justified argument that will help Luke answer Master Yoda's question.

Solution of Problem 6.

Since the number of planets is finite there is at least one planet with the maximal number of connections. Let us call this planet the planet A and let us denote its number of connections by n. Let B_1, B_2,..., B_n be planets connected with the planet A. Since the planets B_1, B_2,..., B_n have a common connection (the planet A), they all have different numbers of connections and their numbers of connections are less than or equal to n, which is the maximal number of connections among all planets. By the Pigeonhole Principle the numbers of connections of the planets B_1, B_2,..., B_n are 1, 2,..., n. So if $n \geq 2$, the there is a planet among them with exactly two connections. To prove that $n \geq 2$ assume that $n = 1$. Since every planet has at least one connection it implies that every planet has exactly one connection. However for an add number of planets it is not possible.