Problem 1. Solve the equation $3^x + 9^x = 27^x$.

Solution:

$$3^x + 3^{2x} = 3^{3x}.$$

Denote: $y = 3^x$, then

$$y + y^2 = y^3.$$
$$y^3 - y^2 - y = 0.$$
$$y(y^2 - y - 1) = 0.$$

Therefore,

$$y = 0 \text{ or } y = \frac{1 \pm \sqrt{5}}{2}.$$

i) $3^x = 0$ has no solutions,
ii) $3^x = \frac{1-\sqrt{5}}{2}$ has no real solutions,
iii) $3^x = \frac{1+\sqrt{5}}{2}$ has a solution

$$x = \log_3\left(\frac{1+\sqrt{5}}{2}\right).$$

Answer: $x = \log_3\left(\frac{1+\sqrt{5}}{2}\right)$.

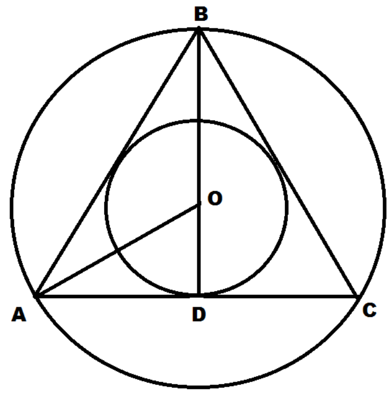

FIGURE 1

Problem 2. An equilateral triangle is inscribed in a circle of area $1m^2$. Then the second circle is inscribed in the triangle. Find the radius of the second circle.

Solution:

In an equilateral triangle the centers of inscribed and circumscribed circles are located at the same point O. AO and BO are bisectors and AD is a height. Therefore AOD is a right triangle and $\angle OAD$ is 60^o. Thus,

$$|OD| = \frac{|AO|}{2}.$$

Since the area of the circumscribed circle is $1m^2$,

$$\pi |AO|^2 = 1$$

and

$$|AO| = \frac{1}{\sqrt{\pi}}.$$

Therefore, $|OD| = \frac{1}{2\sqrt{\pi}}$.

Answer: The radius of the second circle is $\frac{1}{2\sqrt{\pi}}\,m$.

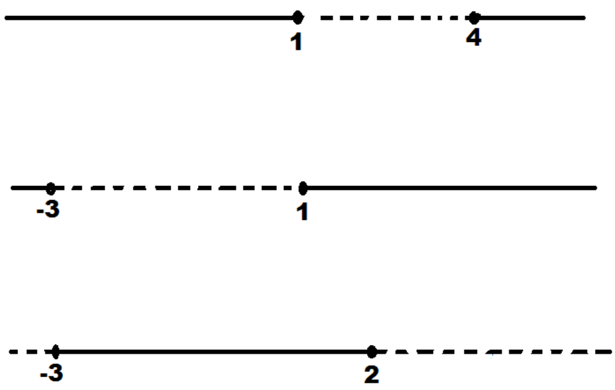

FIGURE 2

Problem 3. Solve the inequality:
$2\sqrt{x^2-5x+4}+3\sqrt{x^2+2x-3}\geq 5\sqrt{6-x-x^2}$.

Solution:

The first square root is defined when $x^2-5x+4\geq 0$, that is $(x-1)(x-4)\geq 0$. Thus, $x\leq 1$ or $x\geq 4$.

The second square root is defined when $x^2+2x-3\geq 0$, that is $(x+3)(x-1)\geq 0$. Thus, $x\leq -3$ or $x\geq 1$.

The third square root is defined when $6-x-x^2\geq 0$, that is $(x+3)(x-2)\leq 0$. Thus, $-3\leq x\leq 2$.

Therefore, all three roots are defined at two points: $x=-3$ and $x=1$. At $x=-3$ the inequality becomes $2\sqrt{28}\geq 0$; and at $x=1$ the inequality becomes $0\geq 5\sqrt{6}$. Thus, only $x=-3$ satisfies the inequality.

Answer: $x=-3$**.**

Problem 4. Peter and John played a game. Peter wrote on a blackboard all integers from 1 to 18 and offered John to choose 8 different integers from this list. To win the game John had to choose 8 integers such that among them the difference between any two is either less than 7 or greater than 11. Can John win the game? Justify your answer.

Solution:

Assume that such integers $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ are chosen and listed in the increasing order. Since all requirements are about differences between integers, we can shift them and assume that $x_1 = 1$. Since the difference between any two integers is either less than 7 or greater than 11, the integers 8,9,10,11,12 cannot be in the list. Consider the following pairs of the remaining integers: (2,13), (3,14), (4,15), (5,16), (6,17), (7,18). The difference between two integers, forming the same pair, is 11. Thus, only one integer from each pair can be represented in the list. So these 6 pairs can produce only six integers in the list. With $x_1 = 1$ it makes 7. Therefore, 8 integers cannot be chosen.

Problem 5. Prove that given 100 different positive integers such that none of them is a multiple of 100, it is always possible to choose several of them such that the last two digits of their sum are zeros.

Solution. Let a_1, a_2, ..., a_{100} be those numbers. Consider the following sums:

$$S_1 = a_1, S_2 = a_1 + a_2, \dots, S_k = a_1 + a_2 + \dots + a_k, \dots,$$
$$S_{100} = a_1 + a_2 + \dots + a_{100}.$$

Consider the last two digits of S_1, S_2,..., S_{100}. If the last two digits of one of these sums are zeros, we are done. Otherwise, there remain only 99 combinations of the last two digits, therefore, two sums, say S_k and S_m, $k < m$, have the same last two digits. Thus the last two digits of $S_m - S_k = a_{k+1} + \dots + a_m$ are zeros.

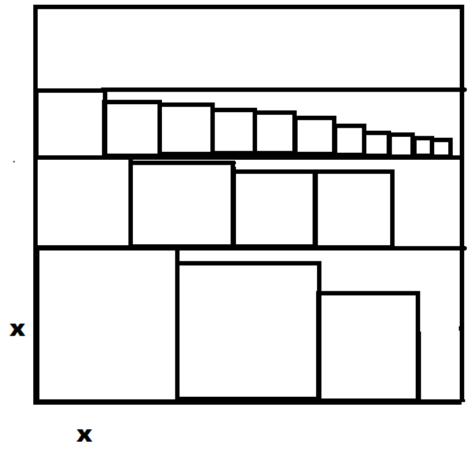

FIGURE 3

Problem 6. Given 100 different squares such that the sum of their areas equals $1/2\ m^2$, is it possible to place them on a square board with area $1\ m^2$ without overlays? Justify your answer.

Solution. First, choose the largest square and place it in the left bottom corner of the board. Let x^2 be the area of this square. Then choose the largest of the remaining squares and place it next to the first one.

Continue the same procedure of choosing the largest of the remaining squares and placing then next to the previous one as it is shown in Figure 3 until there is no room in the first row for the next square. Then draw a horizontal line throw the upper side of the first square and repeat the same procedure above that line. Repeat the same procedure with next rows until all squares are placed. Denote by h_1 the height of the second row by h_2 the height of the third row, and so on. Then the total height of all rows is

$$H = x + h_1 + h_2 + \cdots + h_n.$$

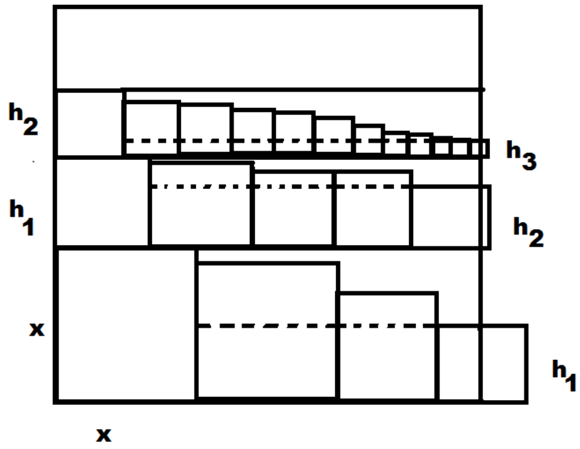

FIGURE 4

Thus, there is enough room for all squares if $H \leq 1$. To prove it, move the first square from the second row to the end of the first row as it is shown in Figure 4. Of course, it will overlap the board. Then, move the first square from the third row to the end of the second row. Apply this procedure to all rows as it is shown in Figure 4.

The sum of the areas of the squares in the first row in Figure 2 is greater than or equal to $x^2+(1-x)h_1$. The sum of the areas of the squares in the second row in Figure 2 is greater than or equal to $(1-h_1)h_2$, and so on. Therefore,

$$\frac{1}{2} \geq x^2 + (1-x)h_1 + (1-h_1)h_2 + \cdots + (1-h_n)h_{n+1}$$

$$\geq x^2 + (1-x)h_1 + (1-x)h_2 + \cdots + (1-x)h_{n+1}$$

$$\geq x^2+(1-x)(h_1+h_2+\cdots+h_{n+1}) \geq x^2+(1-x)(H-x).$$

Therefore

$$x^2 + (1-x)H - x(1-x) \leq \frac{1}{2}.$$

Thus

$$(1-x)H \leq \frac{1}{2} + x - 2x^2 = 1 - x - \left(2x^2 - 2x + \frac{1}{2}\right)$$

and

$$H \leq 1 - \frac{2(x-1/2)^2}{1-x} \leq 1.$$