The Empire Strikes Again but We Have Solutions!

Problem 1. An evil galactic empire is attacking the planet Naboo with numerous automatic drones. The fleet defending the planet consists of 101 ships. By the decision of the commander of the fleet, some of these ships will be used as destroyers equipped with one rocket each or as rocket carriers that will supply destroyers with rockets. Destroyers can shoot rockets so that every rocket destroys one drone. During the attack each carrier will have enough time to provide each destroyer with one rocket but not more. How many destroyers and how many carriers should the commander assign to destroy the maximal number of drones and what is the maximal number of drones that the fleet can destroy?

Problem 1 solution. Let x be the number of destroyers and $101 - x$ be the number of rocket carriers. Thus at the beginning the destroyers can shoot x rockets. Then, every destroyer wil get a rocket from each carrier, that is $x(101 - x)$ additional rockets. So the total number of the rockets the fleet can shoot is:
$N = x + x(101 - x) = -x^2 + 102x = -(x^2 - 102x + 51^2) + 51^2 = -(x-51)^2+2601$. N accepts its maximal value 2601 at $x = 51$. So the commander should assign 51 destroyers and 50 rocket carriers. Thus, the maximal number of drones that the fleet can destroy is 2601.

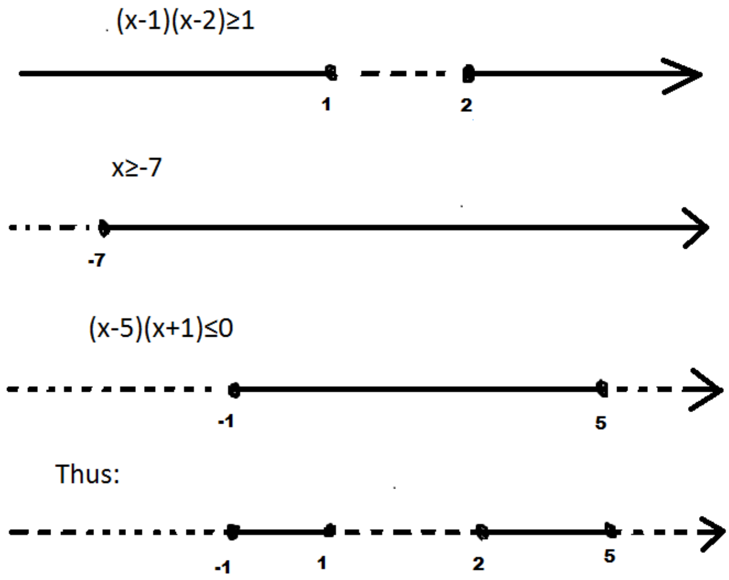

FIGURE 1

Problem 2. Solve the inequality: $\sqrt{x^2-3x+2} \leq \sqrt{x+7}$.

Problem 2 solution. This inequality is equivalent to the following system of inequalities:

$$\begin{cases} x^2-3x+2 \geq 0 \\ x+7 \geq 0 \\ x^2-3x+2 \leq x+7 \end{cases} \leftrightarrow \begin{cases} (x-1)(x-2) \geq 0 \\ x \geq -7 \\ x^2-4x-5 \leq 0 \end{cases}$$

$$\leftrightarrow \begin{cases} (x-1)(x-2) \geq 0 \\ x \geq -7 \\ (x-5)(x+1) \leq 0 \end{cases} \leftrightarrow \begin{cases} x \leq 1 \text{ or } x \geq 2 \\ x \geq -7 \\ -1 \leq x \leq 5 \end{cases}$$

$$\leftrightarrow x \in [-1,1] \cup [2,5]$$

Problem 3. Find all positive real numbers x and y that satisfy the following system of equations:

$$\begin{cases} x^y = y^{x-y} \\ x^x = y^{12y} \end{cases}$$

Problem 3 solution.

$$\begin{cases} y\ln x = (x-y)\ln y \\ x\ln x = 12y\ln y \end{cases} \leftrightarrow \begin{cases} \ln x = \frac{(x-y)}{y}\ln y \\ \frac{x(x-y)}{y}\ln y = 12y\ln y \end{cases}$$

$$\leftrightarrow \begin{cases} \ln x = \frac{(x-y)}{y}\ln y \\ \left(\frac{x(x-y)}{y} - 12y\right)\ln y = 0 \end{cases}$$

From the last equation either $\ln y = 0$ or $\frac{x^2-xy-12y^2}{y} = 0$.

If $\ln y = 0$, then $\ln x = 0$. Therefore, $x = 1$, $y = 1$ is a solution.

If $\frac{x^2-xy-12y^2}{y} = 0$, then $\frac{x^2}{y^2} - \frac{x}{y} - 12 = 0$ and

$\left(\frac{x}{y} - 4\right)\left(\frac{x}{y} + 3\right) = 0.$

Since x and y are positive numbers, $x = 4y$ and

$\ln(4y) = 3\ln(y)$. Therefore, $4y = y^3$, $y = 2$, and $x = 8$.

Thus, the system has two solutions: $\begin{cases} x = 1 \\ y = 1 \end{cases}$ and $\begin{cases} x = 8 \\ y = 2 \end{cases}$

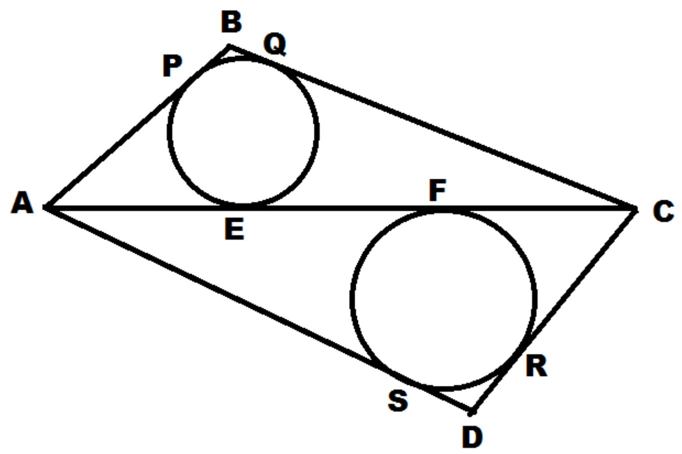

Figure 2

Problem 4. A convex quadrilateral $ABCD$ with sides $AB = 2$, $BC = 8$, $CD = 6$, and $DA = 7$ is divided by a diagonal AC into two triangles. A circle is inscribed in each of the obtained two triangles. These circles touch the diagonal at points E and F. Find the distance between the points E and F.

Problem 4 solution.

Since $|AE| = |AP|$ and $|AF| = |AS|$,

$|EF| = |AF| - |AE| = |AS| - |AP|$.

Since $|CQ| = |CE|$ and $|CF| = |CR|$,

$|EF| = |CE| - |CF| = |CQ| - |CR.|$

Thus, $|EF| = \frac{(|AS|+|CQ|)-(|AP|+|CR|)}{2}$.

Since $|BP| = |BQ|$ and $|DS| = |DR|$,

Thus, $|EF| = \frac{(|AS|+|DS|+|CQ|+|BQ|)-(|AP|+|PB|+|CR|+|DR|)}{2}$.

$= \frac{(|AD|+|BC|)-(|AB|+|CD|)}{2} = \frac{(7+8)-(2+6)}{2} = 3.5$

Problem 5. Find all positive integer solutions n and k of the following equation:

$$\underbrace{11\dots1}_{n}\underbrace{00\dots0}_{2n+3}+\underbrace{77\dots7}_{n+1}\underbrace{00\dots0}_{n+1}+\underbrace{11\dots1}_{n+2}=3k^3.$$

Problem 5 solution.

$\underbrace{11\dots1}_{n}=\frac{99\dots9}{9}=\frac{10^n-1}{9}$

The equation can be rewritten in the following form:

$\frac{10^n-1}{9}10^{2n+3}+7\frac{10^{n+1}-1}{9}10^{n+1}+\frac{10^{n+2}-1}{9}=3k^3$

and then

$10^{3n+3}-10^{2n+3}+7\cdot10^{2n+2}-7\cdot10^{n+1}+7\cdot10^{n+2}-1=27k^3.$

Therefore,

$10^{3n+3}-10\cdot10^{2n+2}+7\cdot10^{2n+2}-7\cdot10^{n+1}+10\cdot10^{n+1}-1=27k^3$

and

$10^{3n+3}-3\cdot10^{2n+2}+3\cdot10^{n+1}-1=27k^3.$

Thus, $(10^{n+1}-1)^3=27k^3$ and $10^{n+1}-1=3k$.

Therefore n can be any positive integer and $k=\underbrace{33\dots3}_{n+1}$.

Problem 6. The Royal Council of the planet Naboo consists of 12 members. Some of these members mutually dislike each other. However, each member of the Council dislikes less than half of the members. The Council holds meetings around the round table. Queen Amidala knows about the relationship between the members so she tries to arrange their seats so that the members that dislike each other are not seated next to each other. But she does not know whether it is possible. Can you help the Queen in arranging the seats? Justify your answer.

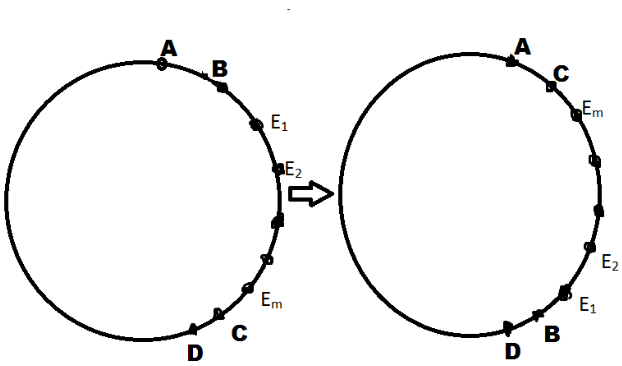

FIGURE 3

Problem 6 solution.

First assume that the members of the Council are seated in an arbitrary order. Let A and B dislike each other (are enemies), but B is seated next to A in the clockwise direction. There are at least 6 members whom A does not dislike, let us cal them friends of A. There are 6 members who are seated next to these friends of A. At least one of them is a friend of B. Indeed, there are not more than 5 members whom B dislike and one of them is A, so there are only 4 remainnig. A himself can be seatteed next to one of his friends. However, there is at least one seat remains that is occupied by a friend of B. So there is a pair C and D such that C is a friend of A, D is a friend of B, and D seats next to C in the clockwise direction. Let E_1, E_2,..., E_m be the members seated between B and C in the clockwise direction. Let Queen Amidala reseat B, E_1, E_2,..., E_m, C in the opposite countelockwise direction from D to A. The only members that changed persons seated next to them are A and D. Now next to A in the clockwise direction is C, whom A likes, and next to D in the counterclockwise direction is B whom D likes. So the number of seating next to each other enemies decreased by one. If there is another pair of enemies seating next to each other then Queen Amidala repeats the procedure. Therefore in a finite number of steps all such pairs will be reseated.