Bilbo's New Adventures

Problem 1. Solve the equation:
$\sqrt{x}+\sqrt{x+1}-\sqrt{x+2}=0.$

Solution.

The left side of the equation is defined for $x \geq 0$.
Moving $\sqrt{x+2}$ to the right hand side of the equation one gets:
$\sqrt{x}+\sqrt{x+1}=\sqrt{x+2}.$
Since both sides are nonnegative this equation is equivalent to
$(\sqrt{x}+\sqrt{x+1})^2=x+2.$
Therefore
$x+2\sqrt{x(x+1)}+x+1=x+2,$
$2\sqrt{x(x+1)}=1-x.$
This equation is equivalent to

$$\begin{cases} 4x^2+4x=1-2x+x^2 \\ x \leq 1 \end{cases}.$$

Thus, $3x^2+6x-1=0$, and
$x=\frac{-6\pm\sqrt{36+12}}{6}=\frac{-6\pm4\sqrt{3}}{6}=-1\pm\frac{2\sqrt{3}}{3}.$
Since $x>0$, $x=-1+\frac{2\sqrt{3}}{3}\leq 1.$

Answer: $x=-1+\frac{2\sqrt{3}}{3}$

Problem 2. Solve the inequality: $\ln(x^2 + 3x + 2) \leq 0$.

Solution. This inequality is equivalent to the following double inequality:

$0 < x^2 + 3x + 2 \leq 1$.

Thus $\begin{cases} x^2 + 3x + 2 > 0 \\ x^2 + 3x + 1 \leq 0 \end{cases}$.

i) $(x + 1)(x + 2) > 0$.

Therefore, $x < -2$ or $x > -1$.

ii) $x^2 + 3x + 1 \leq 0$.

$x^2 + 3x + 1 = 0$,

$x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$.

Therefore, $\frac{-3-\sqrt{5}}{2} \leq x \leq \frac{-3+\sqrt{5}}{2}$.

Since $\frac{-3-\sqrt{5}}{2} < -2$ and $\frac{-3+\sqrt{5}}{2} > -1$, one gets:

either $\frac{-3-\sqrt{5}}{2} \leq x < -2 \quad \text{or} \quad -1 < x < \frac{-3+\sqrt{5}}{2}$.

Answer: $\left[\frac{-3-\sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3+\sqrt{5}}{2}\right]$.

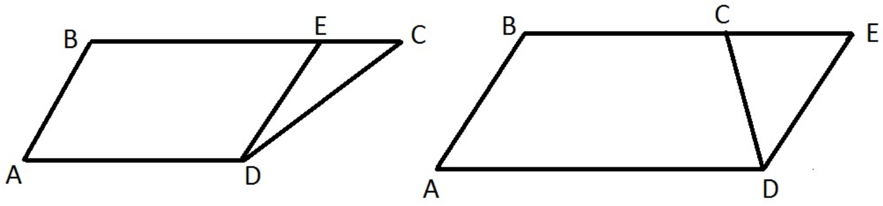

FIGURE 1

Problem 3. In the trapezoid $ABCD$ ($AD \| BC$) $|AD|+|AB| = |BC|+|CD|$. Find the ratio of the length of the sides AB and CD ($|AB|/|CD|$).

Solution. Let DE be parallel to AB. The point E may be to the left from the point C, or to the right, or both points may coinside.
First case (the left picture). The point E is to the left from the point C. Since $ABED$ is a parallelogram,

$|BE| = |AD|$, $|AB| = |DE|$.

Given,
$|AD| + |AB| = |BC| + |CD|$,

one gets

$|BE| + |DE| = |BC| + |CD| = |BE| + |EC| + |CD|$.

Therefore,

$|DE| = |EC| + |CD|$.

But this cannot happen since in a triangle (in our case the triangle CDE) the length of every side is strictly less than the sum of the lengths of the other two sides.

Second case (the right picture). The point E is to the right from the point C. Then

$|AD| + |AB| = |BC| + |CD| = |BE| - |CE| + |CD| = |AD| - |CE| + |CD|$.

Therefore,

$|DE| = |AB| = -|CE| + |CD|$

and

$|CD| = |DE| + |CE|$.

This cannot happen either.

So, the points C and E coinside, and $\frac{|AB|}{|CD|} = 1$.

Problem 4. Gollum gave Bilbo a new riddle. He put 64 stones that are either white or black on an 8×8 chess board (one piece per each of 64 squares). At every move Bilbo can replace all stones of any horizontal or vertical row by stones of the opposite color (white by black and black by white). Bilbo can make as many moves as he needs. Bilbo needs to get a position when in every horizontal and in every vertical row the number of white stones is greater than or equal to the number of black stones. Can Bilbo solve the riddle and what should be his solution?

Solution. At each move Bilbo should find any horizontal or vertical row where the number of black stones is greater than the number of white stones and replace all stones of that row by stones of the opposite color. Then, after each move the number of white stones will encrease. Since the maximal number of white stones on the board cannot exceed 64, after some number of moves it will reach its maximum. And in the position where the number of white stones is maximal in every horizontal and in every vertical row the number of white stones is greater than or equal to the number of black stones. Indeed, if there exists at least one horizontal or vertiacal row where the number of black stones is greater than the number of white stones, then by replacing all stones of that row by stones of the opposite color one will increase the number of white stones on the board. This would exceed the maximum which is impossible.

Problem 5. Two trolls Tom and Bert caught Bilbo and offered him a game. Each player got a bag with white, yellow, and black stones. The game started with Tom putting some number of stones from his bag on the table, then Bert added some number of stones from his bag, and then Bilbo added some stones from his bag. After that three players were making moves. At each move a player chooses two stones of different colors, takes them away from the table, and puts on the table a stone of the color different from the colors of chosen stones. Game ends when stones of one color only remain on the table. If the remaining stones are white Tom wins and eats Bilbo, if they are yellow, Bert wins and eats Bilbo, if they are black, Bilbo wins and is set free. Can you help Bilbo to save his life by offering him a winning strategy?

Solution. Bilbo should add stones to make the number of black stones odd, and numbers or white and yellow stones both even (or to make the number of black stones even, and numbers or white and yellow stones both odd). Thus the numbers of white and yellow stones will have the same parity and the number of black stones a different parity. After each move of each player the parities of the numbers of white, yellow, and black stones change. So how would the players play does not matter, the numbers of white and yellow stones will have the same parity and the number of black stones a different parity. At the final position the numbers of stones of two colors are zeros, that is the same parity, thus these numbers are the numbers of white and yellow stones. Therefore, the stones remaining on the table are black. Bilbo wins!

Problem 6. There are four roads in Mirkwood that are straight lines. Bilbo, Gandalf, Legolas, and Thorin were travelling along these roads, each along a different road, at a different constant speed. During their trips Bilbo met Gandalf, and both Bilbo and Gandalf met Legolas and Thorin, but neither three of them met at the same time. When meeting they did not stop and did not change the road, the speed, and the direction. Did Legolas meet Thorin? Justify your answer.

Solution. Let $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, and $\mathbf{v_4}$ be the velocities of Bilbo, Gandalf, Legolas, and Thorin respectively. Consider the moving coordinate system with the origin at the Bilbo's position. In this coordinate system Bilbo does not move but the roads move with the velocity $-\mathbf{v_1}$. Therefore, in this moving coordinate system Gandalf, Legolas, and Thorin move with the velocity vectors $\mathbf{v_2}$-$\mathbf{v_1}$, $\mathbf{v_3}$-$\mathbf{v_1}$, and $\mathbf{v_4}$-$\mathbf{v_1}$ respectively. Thus they move along the straight lines. (This follows from Galileo's principle in Physics as well. Also it can be proved using similar triangles.) Also, in the moving coordinate system they still move at different constant speeds. Since Gandalf, Legolas, and Thorin met Bilbo, the straight lines along which they move intersect at the Bilbo's location. Gandalf met Legolas so their straigt lines intersect, but since neither three of them met at the same time their straight lines have another point of intersection, and, therefore, Gandalf and Legolas move along the same line. Similarly, one can prove that Gandalf and Thorin move along the same line. Thus all three of them, Gandalf, Legolas, and Thorin move along the same straight line. Since Legolas and Thorin have different speeds they must meet.

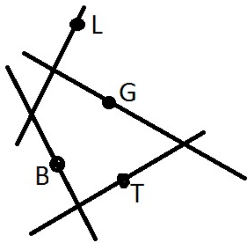

Fig. 1

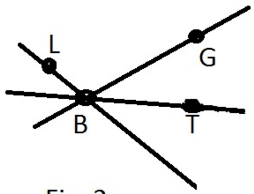

Fig. 2

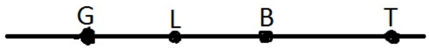

Fig. 3