KETTERING XX MATHEMATICS OLYMPIAD

Star Wars Continued

Problem 1. Darth Vader urgently needed a new Death Star battle station. He sent requests to four planets asking how much time they would need to build it. The Mandalorians answered that they can build it in one year, the Sorganians in one and a half year, the Nevarroins in two years, and the Klatooinians in three years. To expedite the work Darth Vader decided to hire all of them to work together. The Rebels need to know when the Death Star is operational. Can you help the Rebels and find the number of days needed if all four planets work together? We assume that one year=365 days.

Solution of Problem 1. Denote by $x$ the total amount of work. In one day the Mandalorians complete $\frac{x}{365}$, the Sorganians $\frac{2}{3} \cdot \frac{x}{365}$, the Nevarroins $\frac{1}{2} \cdot \frac{x}{365}$, and the Klatooinians $\frac{1}{3} \cdot \frac{x}{365}$. Thus, together in one day they complete

$$\frac{x}{365} + \frac{2}{3} \cdot \frac{x}{365} + \frac{1}{2} \cdot \frac{x}{365} + \frac{1}{3} \cdot \frac{x}{365} = \frac{x}{365} \left(1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{5}{2} \cdot \frac{x}{365} = \frac{x}{146}.$$ 

Therefore, working together they will build the battle station in 146 days.
Problem 2. Solve the inequality: $(\sin \frac{\pi}{12})^{\sqrt{1-x}} > (\sin \frac{\pi}{12})^x$.

Solution of Problem 2.

$0 < \sin \frac{\pi}{12} < 1$ implies $\sqrt{1-x} < x$

\[
\begin{cases}
1 - x \geq 0 \\
x > 0 \\
1 - x < x^2
\end{cases}
\]

\[
\begin{cases}
0 < x \leq 1 \\
x^2 + x - 1 > 0
\end{cases}
\]

$x^2 + x - 1 = 0$

$x = \frac{-1 \pm \sqrt{5}}{2}$

\[
\begin{cases}
0 < x \leq 1 \\
x < \frac{-1-\sqrt{5}}{2} \\
x > \frac{-1+\sqrt{5}}{2}
\end{cases}
\]

$\frac{-1 + \sqrt{5}}{2} < x \leq 1$.

Answer: $\left(\frac{\sqrt{5}-1}{2}, 1\right]$. 
Problem 3. Solve the equation:

\[ \sqrt{x^2 + 4x + 4} = x^2 + 3x - 6 \]

Solution of Problem 3.

\[ \sqrt{x^2 + 4x + 4} = \sqrt{(x + 2)^2} = |x + 2| \]

\[ |x + 2| = x^2 + 3x - 6 \]

Case I. \( x < -2 \)

\[ |x + 2| = -x - 2 \]
\[ -x - 2 = x^2 + 3x - 6 \]
\[ x^2 + 4x - 4 = 0 \]
\[ x = -2 \pm \sqrt{8} = -2 \pm 2\sqrt{2} \]

\[ \begin{cases} x < -2 \\ x = -2 \pm 2\sqrt{2} \\ x = -2 - 2\sqrt{2} \end{cases} \]

Case II. \( x \geq -2 \)

\[ |x + 2| = x + 2 \]
\[ x + 2 = x^2 + 3x - 6 \]
\[ x^2 + 2x - 8 = 0 \]
\[ x = -1 \pm \sqrt{9} = -1 \pm 3 \]

\[ \begin{cases} x \geq -2 \\ x = -1 \pm 3 \\ x = 2 \end{cases} \]

Answer: \( \{-2 - 2\sqrt{2}, 2\} \).
Problem 4. Solve the system of inequalities on $[0, 2\pi]$:
\[
\begin{align*}
\sin(2x) & \geq \sin(x) \\
\cos(2x) & \leq \cos(x)
\end{align*}
\]

Solution of Problem 4.

1. $\sin(2x) \geq \sin(x)$

\[
2 \sin x \cos x \geq \sin x \\
\sin x(2 \cos x - 1) \geq 0
\]

Let us consider three cases: $\sin x = 0$, $\sin x > 0$, $\sin x < 0$.

I case. $\sin x = 0$.
In $[0, 2\pi]$ there are three solutions $x = 0$, $x = \pi$, $x = 2\pi$.
II case. $\sin x > 0$, that is $0 < x < \pi$. Then

\[
\cos x \geq \frac{1}{2}
\]

In $(0, \pi)$ this implies that $0 < x \leq \frac{\pi}{3}$.
III case. $\sin x < 0$, that is $\pi < x < 2\pi$. Then

\[
\cos x \leq \frac{1}{2}
\]

In $(\pi, 2\pi)$ this implies that $\pi < x \leq \frac{5\pi}{3}$.

Combining all three cases, one gets
\[
\begin{cases}
0 \leq x \leq \frac{\pi}{3} \\
\pi \leq x \leq \frac{5\pi}{3} \\
x = 2\pi
\end{cases}
\]

2. $\cos(2x) \leq \cos(x)$

\[
\begin{align*}
\cos^2 x - \sin^2 x & \leq \cos x \\
\cos^2 x - 1 + \cos^2 x & \leq \cos x \\
2 \cos^2 x - \cos x - 1 & \leq 0
\end{align*}
\]

Denote $\cos x = t$, then
\[
2t^2 - t - 1 \leq 0 \\
t = \frac{1 \pm \sqrt{1 + 8}}{4} = \frac{1 \pm 3}{4}
\]
\[ t = -\frac{1}{2} \text{ or } t = 1 \]
\[ -\frac{1}{2} \leq t \leq 1 \]
\[ -\frac{1}{2} \leq \cos x \leq 1 \]
\[ 0 \leq x \leq \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \leq x \leq 2\pi \]

Answer: \[ [0, \frac{\pi}{3}] \cup \left[\frac{4\pi}{3}, \frac{5\pi}{3}\right] \cup \{2\pi\} \]
**Problem 5.** The planet Naboo is under attack by the imperial forces. Three rebellian camps are located at the vertices of a triangle. The roads connecting the camps are along the sides of the triangle. The length of the first road is less than or equal to 20 miles, the length of the second road is less than or equal to 30 miles, and the length of the third road is less than or equal to 45 miles. The Rebels have to cover the area of this triangle by a defensive field. What is the maximal area that they may need to cover?

**Solution of Problem 5.** Denote the vertices of the triangle \(A, B,\) and \(C.\) Then, \(|AB| \leq 20, |AC| \leq 30, |BC| \leq 45.\) Let \(\theta\) be the angle between \(AB\) and \(AC\) and \(S\) be the area of the triangle \(ABC.\)

\[
S = \frac{|AB| \cdot |AC|}{2} \sin \theta \leq \frac{|AB| \cdot |AC|}{2} \leq \frac{20 \cdot 30}{2} = 300.
\]

So, the largest area is the area of the right triangle with sides 20, 30, and

\[
\sqrt{20^2 + 30^2} = \sqrt{1300} = 10\sqrt{13} \leq 10 \cdot 4 = 40 < 45.
\]

Answer: The maximal area is 300 square miles.
Problem 6. The Lake Country on the planet Naboo has the shape of a square. There are nine roads in the country. Each of the roads is a straight line that divides the country into two trapezoidal parts such that the ratio of the areas of these parts is 2:5. Prove that at least three of these roads intersect at one point.

Solution of Problem 6.

Let us consider a road $KL$ that intersects sides $AD$ and $BC$. It divides the square on two trapezoids $AKLB$ and $DKLC$ the areas of which are related as 2:5. The area of the trapezoid equals to the product of lengths of the middle line and the height. the hights of the trapezoids $DKLC$ and $AKLB$ are equal. Therefore the middle lines of triapesoids $FM$ and $MH$ are related as 2:5. Consider four points that divides $FH$ and $EG$ in relation 2:5 and 5:2. Then each road should pass through one of this four points. Since there are nine roads, at least three roads should pass through one of these points.