MOVIES, GAMBLING, SECRET CODES, JUST MATRIX MAGIC

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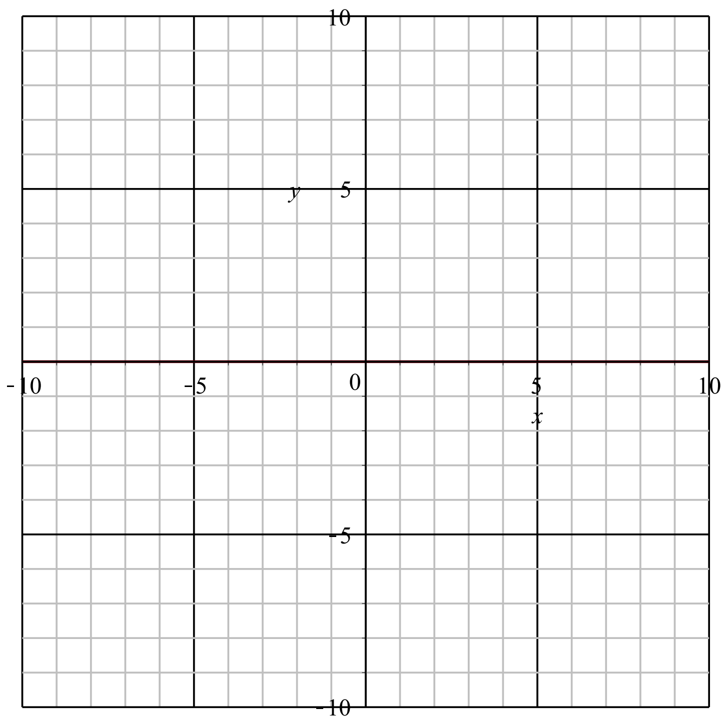
1. The Cartesian Coordinate System

In the Cartesian system points are defined by giving their coordinates. Plot the following points: $(2, 1)$, $(-1, 3)$, $(-2, -2)$, $(2, -1)$.

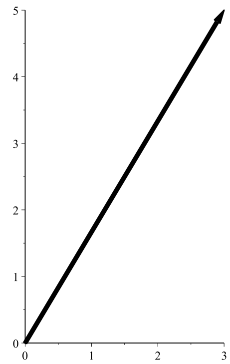


2. What is a Matrix?

A matrix is simply an array of numbers. Here are some examples of matrices.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

What do matrices do? They transform the space. Here is how. Plot an arrow, in math we call it vector, connecting points $(0, 0)$ and $(3, 5)$.

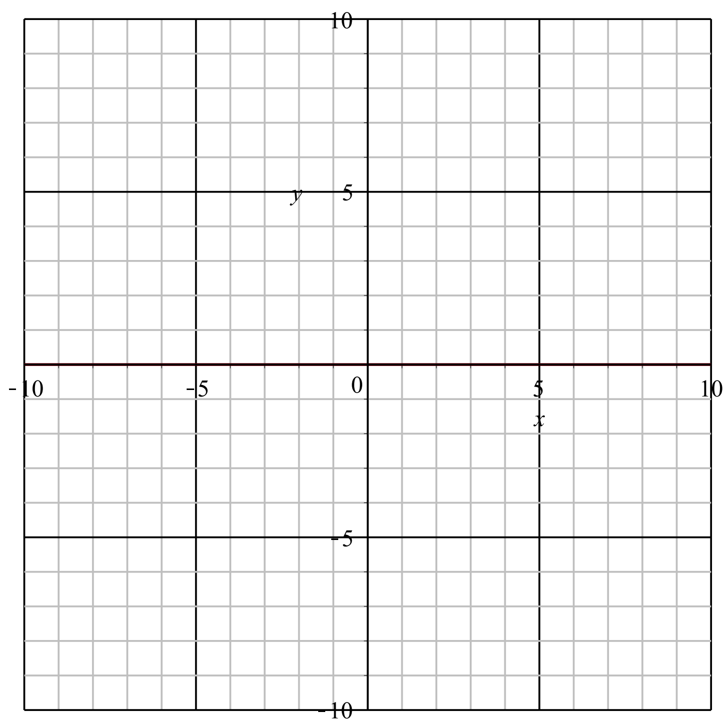


Call this vector v. Its coordinates are calculated by subtracting the starting point from the end point

$$v = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Now plot arrows with the following end points to create the letter "F":

(1) from $(0,0)$ to $(0,5)$
(2) from $(0,5)$ to $(3,5)$
(3) from $(0,3)$ to $(2,3)$



What vectors can you identify on this picture?

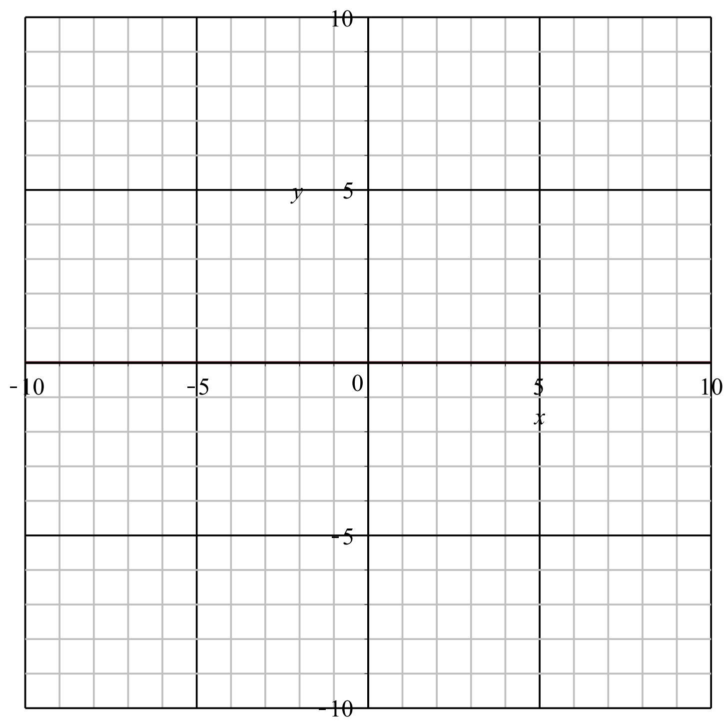
Let's see what the matrix A does to a vector v. Compute the action of matrix A on vector v.

$$A \cdot v = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (1)(1) + (2)(1) \\ (3)(1) + (4)(1) \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

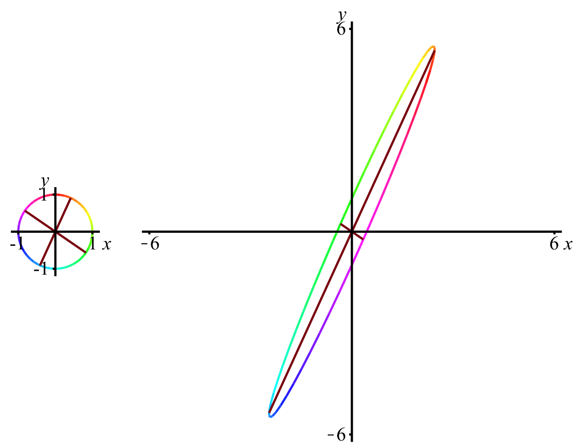
In general, we multiply matrix by a vector as follows:

$$A \cdot v = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} a_{11}v_x + a_{12}v_y \\ a_{21}v_x + a_{22}v_y \end{pmatrix}$$

Try some other vectors to see what the matrix A will do to them.



Here is a graph of a transformed circle, the image is an ellipse. To produce this graph in Maple click: Tools → Tutors → Linear Algebra → Linear Transform Plot, then Edit Matrix button, edit your matrix, and click Close button.



3. In the footsteps of Steven Spielberg

Let's now play with Maple software and create an animation. Instructions will be provided during session.

To understand how the rotation matrix works, we choose the rotation angle of 90 degrees or $\pi/2$ to obtain

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Now apply this matrix to vector $v =< 1, 1 >$. Try some other vectors too.

Try other angles, $\theta = \pi/4$ or $\theta = \pi/3$.

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ -\sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 0.5 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 0.5 \end{pmatrix}$$

Now apply this matrix to vector $v =< 1, \sqrt{3} >$.
We are now ready to play our animation.

4. Making more complex animations by using Matrix Algebra

To make more complex animations we need to compute expressions of the type $A \cdot B \cdot v$, application of two (or more) matrices to a vector. This can be done in two ways

$$A \cdot (B \cdot v) \quad \text{or} \quad (A \cdot B) \cdot v.$$

Try the first method first and compute

$$A \cdot (B \cdot v) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) =$$

To try the second method we need to learn ho to multiply matrices. This is simple, to compute $A \cdot B$, just multiply matrix A by each column of matrix B (they are column vectors) and arrange the results in a matrix.

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} , \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 5 & 1 \\ 11 & 1 \end{pmatrix}$$

Then

$$(A \cdot B) \cdot v = \begin{pmatrix} 5 & 1 \\ 11 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$$

should give the same result.

Let's now play with Maple to put two transformations (matrices) at work together. Which multiplication method will be faster for moving many vectors at the same time?

5. Surprises with Matrix Algebra

First let's add (subtraction goes the same) two matrices, it is easy, just add them by adding corresponding entries:

$$A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+(-1) \\ 3+2 & 4+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 5 \end{pmatrix}$$

Most laws of matrix multiplications and additions/subtractions are just like they are for numbers.

- Associative laws

$$(A \cdot B) \cdot C = A \cdot (B \cdot C) \quad \text{and} \quad (A \pm B) \pm C = A \pm (B \pm C)$$

- Distributive laws

$$(A \pm B) \cdot C = A \cdot C \pm B \cdot C \quad \text{and } A \cdot (B \pm C) = A \cdot B \pm A \cdot C$$

- Commutative laws (surprise!)

$$A \pm B = B \pm A \quad \text{but it may happen that } A \cdot B \neq B \cdot A$$

Problem 1. Find examples of two matrices A and B such that $A \cdot B \neq B \cdot A$.

Problem 2. What does your matrix A do to vectors? What does your matrix B do to vectors? Do you see why $A \cdot B \neq B \cdot A$?

Problem 3. Find example of two matrices A and B, having none of their entries equal 0, but such that $A \cdot B = 0$.

Problem 4. What does your matrix A do to vectors? What does your matrix B do to vectors? Do you see why $A \cdot B = 0$?

Problem 5. It gets better, if x is a number, and if $x^2 = 0$, then $x = 0$. But... find a matrix A who's no entry equals zero, but $A \cdot A = 0$.

problem 6. What does your matrix A do to vectors? Do you see why $A \cdot A = 0$?

Problem 7. Explain the importance of the Associative Law for making animations.

6. Gambler's Ruin Problem

Imagine that:

(1) You are in Vegas and you have \$20 but need \$50 to pay for the hotel. You decide to gamble in a casino. The only possible bet you can make is a bet of \$10, and then a fair coin is tossed. If "heads" results, you win \$10 (you get your \$10 back plus an additional \$10), and if its "tails", you lose the \$10 bet. The coin being fair has exactly the same chances of coming up "heads" or "tails". What is the chance that you will pay for the hotel? What if you had started with \$10? or with \$30, \$40?
(2) What if you had \$40 and could bet either \$10 each time or \$20 each time and you needed \$100? Which strategy is better?
(3) What if the chances were not $50-50$? A roulette wheel in the US has 18 red numbers, 18 black numbers and 2 green numbers. Winning a red or black bet doubles your money and you lose it all if you lose. A red or black bet has a probability of $18/(18+18+2) = 18/38 = 9/19$ of winning on each spin of the wheel.

How can we use matrices to help us out? At any stage in the process, there are six possible states, depending on your fortune. You have

(1) \$0 (and you have lost)
(2) \$10 (and you are still playing)
(3) \$20 (and you are still playing)
(4) \$30 (and you are still playing)
(5) \$40 (and you are still playing)
(6) \$50 (and you have won)

You know that when you begin playing, you are in a certain state (having \$20 in the case of the very first problem). After you have played a while, lots of different things could have happened, so depending on how long you have been going, you have various probabilities of being in the various states. Chances of being in each state after the first game can be described using matrix multiplication:

$$T \cdot p = \begin{pmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_{10} \\ p_{20} \\ p_{30} \\ p_{40} \\ p_{50} \end{pmatrix} = \begin{pmatrix} p_0 + 0.5p_{10} \\ 0.5p_{20} \\ 0.5p_{10} + 0.5p_{30} \\ 0.5p_{20} + 0.5p_{40} \\ 0.5p_{30} \\ 0.5p_{40} + p_{50} \end{pmatrix}$$

If you've played $2, 3, ..., 200$, times, your chances to be in any of the six states are equal

$$T^2 \cdot p,\ T^3 \cdot p,\ T^{200} \cdot p,$$

Let's use Maple to see what happens, when we start from

$$\begin{pmatrix} p_0 \\ p_{10} \\ p_{20} \\ p_{30} \\ p_{40} \\ p_{50} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Use Maple to solve the remaining questions.

7. Cryptography

Before we get to encrypting and decrypting messages let's find out how we can divide matrices. Well, in math we don't call it that way. But..., with numbers, what does it really mean to divide a by b?

$$\frac{a}{b} = a \cdot b^{-1},$$

so we can say that division is the same as multiplication by an inverse. Same with matrices. For any given matrix A we may attempt to find its inverse A^{-1}.

Problem 1. How should we define A^{-1}?.

Problem 2. Find a matrix A whose none of the entries is zero, but whose inverse A^{-1} does not exist.

Problem 3. What does your matrix A do to vectors?

Problem 4. Find a formula for an inverse matrix to

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

To encrypt a message using matrices, we first assign numbers $1 - 26$ to letters.

Blank	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

The message: "SECRET CODE" corresponds to a sequence 19 5 3 18 5 20 0 3 15 4 5.
Let's use an **encoding matrix**

$$E = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$$

and the **plain text matrix**

$$T = \begin{pmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{pmatrix}$$

We have used Blank or 0 at the end to make the columns come out even.

Problem 5. Encrypt the message by multiplying matrices $A \cdot B$ and arranging letters, column by column, in one row.

Problem 6. Find the inverse matrix E^{-1}, the **decoding matrix**. How can you use it to decode the message?

Problem 7. Use Maple to encrypt and decrypt messages.

Activity Encrypt a message and give it to your neighbour together with your encoding matrix. Your neighbour needs to decrypt your message.

8. Systems of linear equations

How about just one equation first. How do we solve $x + 3 = 7x - 9$?
Problem 1. Write the following equation using matrix algebra

$$\begin{cases} x + 2y = 1 \\ x + 3y = 3 \end{cases}$$

Problem 2. Solve the above system of equations using matrices.

Problem 3. A concert hall has 10,000 seats and two categories of ticket prices, \$25 and \$35. Since Lady Gaga was performing the concert was sold out. How many tickets in each category were sold if the return was $\$300,000$? $\$325,000$? Could the return be $\$400,000$?

Problem 4. Solve

$$\begin{cases} 3x + 3y + 6z + 5w = 10 \\ 4x + 5y + 8z + 2w = 15 \\ 3x + 6y + 7z + 4w = 30 \\ 4x + y + 6z + 3w = 25 \end{cases}$$

See you next time!!!

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