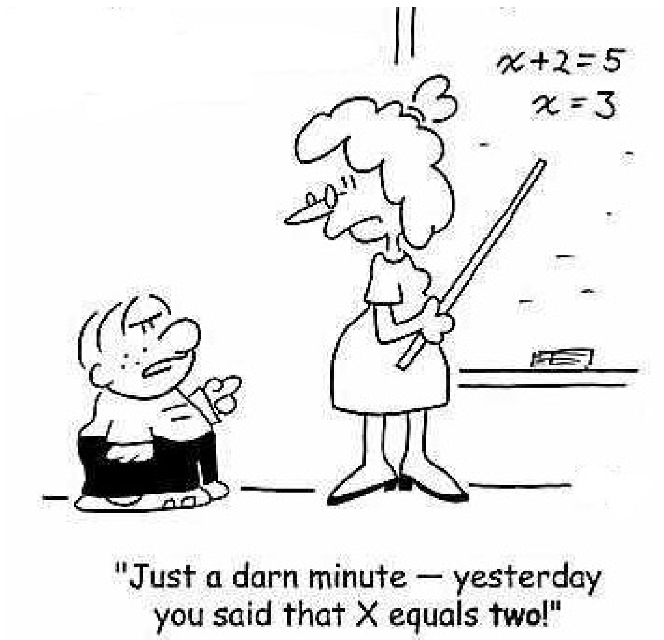

Mysteries of Algebraic Equations.

Formulas.

Identities.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Linear equations.

$$ax = b$$

i) $a \neq 0 \Rightarrow$ a unique solution $x = \frac{b}{a}$,
ii) $a = 0,\ b \neq 0 \Rightarrow$ no solutions,
iii) $a = 0,\ b = 0 \Rightarrow$ infinitely many solutions (every number is a solution).

Quadratic equations.

$$ax^2 + bx + c = 0, \quad a \neq 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

i) $b^2 - 4ac > 0 \Rightarrow$ two solutions $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
ii) $b^2 - 4ac = 0 \Rightarrow$ one (double) solution $x_{1,2} = \frac{-b}{2a}$,
iii) $b^2 - 4ac < 0 \Rightarrow$ no real solutions.

Let $b^2 - 4ac > 0$ and x_1, x_2 are solutions, then
i) $ax^2 + bx + c = a(x - x_1)(x - x_2)$,
ii) $x_1 + x_2 = -\frac{b}{a}$, $x_1 x_2 = \frac{c}{a}$ (Viéte's theorem).

Cubic equations.

$$x^3 + px + q = 0$$

$$(a+b)^3 = 3ab(a+b) + a^3 + b^3$$

$$3ab = -p, \quad a^3 + b^3 = -q$$

Fourth degree equation.

$$x^4 + px^2 + qx + r = 0$$

$$x^4 + 2zx^2 + z^2 = (2z-p)x^2 - qx + (z^2 - r)$$

$$2\sqrt{2z-5}\sqrt{z^2-r} = -q$$

$$(2z-5)(z^2-r) = \frac{q^2}{4}$$

Problems.

Problem 1. Represent $x^2 + 4x + 4$ as a perfect square.

Problem 2. Represent $4x^2 + 20x + 25$ as a perfect square.

Problem 3. Represent $x^3 - 3x^2 + 3x - 1$ as a perfect cube.

Problem 4. Represent $x^3 + 6x^2 + 12x + 8$ as a perfect cube.

Problem 5. Factor $9x^2 - 49$.

Problem 6. Factor $x^3 + 1$.

Problem 7. Factor $8x^3 - 27$.

Problem 8. Simplify $\frac{x^3+125}{x+5} + \frac{x^2-4}{x-2}$.

Problem 9. Find values of a such that the equation

$$(a^2-1)x = a+1$$

(i) has a unique solution,
(ii) has no solutions,
(i) has infinitely many solutions.

Problem 10. Solve the quadratic equation $x^2 + 2x - 4 = 0$.

Problem 11. Solve the quadratic equation $3x^2 + x - 3 = 0$.

Problem 12. Factor $x^2 + 2x - 4$.

Problem 13. Factor $3x^2 + x - 3$.

Problem 14. Find values of a such that the equation

$$x^2 + 2ax + a = 0$$

(i) has two different solutions,
(ii) has one (double) solution,
(iii) has no solutions.

Problem 15. Without solving the equation $x^2 + 2ax + a = 0$ find
i) the sum of the solutions,
ii) the product of the solutions.

Problem 16. Find a quadratic equation with solutions $x_1 = 2$ and $x_2 = -3$.

Problem 17. Find a solution to the cubic equation $x^3 - 6x - 9 = 0$.

Problem 18. Find a solution to the cubic equation $x^3 + 9x - 26 = 0$.

Problem 19. Find a solution to the cubic equation $x^3 + x - 6 = 0$.

Problem 20. Find a solution to the cubic equation $x^3 + 6x^2 + 9x - 2 = 0$.

Problem 21. Solve $x^4 - 8x + 6 = 0$.

Problem 22. Solve $x^4 + 8x^3 + 24x^2 - 112x + 52 = 0$.

Problem 23. Solve $x^4 - 15x^2 - 10x + 24 = 0$.

References.
1. J. Bewersdorff, Galois Theory for Beginners, AMS, 2006.