The Mathematics of Light

Math Club Topics
Overview

- Review of fractions
- Review of first degree equations
- Review of functions
- The sine function
- Electromagnetic waves, light, colors and the wavelength of light
- Geometrical optics: light rays, refraction and reflection of light
- Lenses, image formation through lenses and the thin lens equation
- Optical instruments with lenses.

The Mathematics of Light
Ms. Gina Rablau, Kettering University
Review of functions

- In general, a function is a black box which is promised to do specific task(s), each and every time we ask it to do.

- You give something to the black box as *input*, and it returns the result as *output*, as shown by the schematic function diagram above.
The “Clean” Function

- A washing machine can be regarded as a black box that performs the function “to clean”
Unlike the washing machine, a mathematical function is a rule (the promised “to do” list) that assigns to each real number $x$ another real number $y$.

The input to the function, $x$, is called the argument of the function. The output of the function, $y$, is called the value of the function.

In other words, a mathematical function is a black box that transforms a set of numbers into another set of numbers, according to a specific rule.
Example: The Doubling Function

- A simple example of a function: *the doubling function*

\[ f(x) = 2x \]

- For any value of the argument, \( x_0 \), the value of the function is twice the value of the argument, \( y = 2x_0 \).

- The input of the function can be any real number, and the output will be twice that number.
Graph of a Function

- The graph of a function is a powerful graphical way of representing what a mathematical function does.

- The graph of a function shows all the possible values of the input to (argument of) that function, and all the corresponding values of the output of that function.

- Example: the graph of the doubling function

![Graph of the doubling function](image)
The Sine Function

- A particular class (or family) of functions of huge importance for many practical applications, including for Light and Optics, is the class of **trigonometric functions**.

- The input of a trigonometric function is **an angle**. Angles are measured in **degrees** (which have the symbol °).

- Another unit used for angles in science and engineering is the **radian** (which has the symbol rad). There is a simple relation between these two units:

  \[ 180° = \pi \text{ radians} \]

  where \( \pi = 3.14.. \)
The action of sine function can be understood from its graph.
So how does a sine function actually do its black box magic?

$$\sin A = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} = \frac{a}{c}$$

In a right angle triangle, the sine of an angle is equal to the ratio between the length of the side of the triangle opposite that angle and the length of the hypotenuse:
Class Activity

- Step 1: Please draw a right triangle with an acute angle $A$ and measure the length of the opposite side of angle $A$ and the length of the hypotenuse and evaluate the ratio and record it.
- Step 2: Now, use a rapporteur and measure angle $A$ and compute $\sin A$. Please compare this result with the one found in Step 1.
- Step 3: Please extend both the base and the hypotenuse of your triangle obtaining in this way a new triangle. Repeat Step 1 and 2 on this new triangle. Did you obtain a different value for $\sin A$ when using the measurements of this new triangle? Is the result consistent with your expectation?
Web Resources for Optics

- **HyperPhysics – Light and Vision** (Georgia State University)
  - [http://hyperphysics.phy-astr.gsu.edu/hbase/hph.html#ahph](http://hyperphysics.phy-astr.gsu.edu/hbase/hph.html#ahph)

- **Molecular Expressions – Physics of Light and Color** (FSU and the National High Magnetic Fields Lab)

- **General Physics Java Applets** by Surendranath Reddy
  - [http://surendranath.tripod.com/](http://surendranath.tripod.com/)

- Optics is by its nature a visual science. A wealth of Java Applets found on these sites help visualize optics phenomena.
Optics is the science that studies light.

Light, in its broadest sense, is more than only visible light.

The ultraviolet (UV) and infrared (IR) regions of the spectrum facilitate some of the most important applications of optics.
Wave Properties of Light

- Light travels as a wave, similar to water waves
  
  ![Wave diagram]

  or to the waves on the vibrating strings of a guitar

- But, unlike these waves, the oscillations of a light wave are in **electric field strength**, not in spatial displacement.
Wave Properties of Light

What oscillates and propagates in a light wave is an electric field.

Snapshot (in time) of the electric field of the wave
Each “color” of light has a number associated to it – the wavelength ($\lambda$) of the light wave (measured in nm or $\mu$m).

Visible light extends from 400 nm (violet) to ~800 (Red) nm.

The spatial profile of the wave repeats itself identically after a distance equal to the wavelength – hence the name.

The wavelength acts as a length scale built into the light wave.
The Electromagnetic Spectrum

- Light is a tiny part of a huge spectrum of electromagnetic waves

Light in vacuum has **wavelengths** between approximately 400 nm and 800 nm; 1 nm $\equiv 10^{-9}$ m (a nanometer)

1 $\mu$m $\equiv 10^{-6}$ m (a micron); 1 Å $\equiv 10^{-10}$ m $\equiv 0.1$ nm
In a homogeneous medium (a medium that has the same index of refraction everywhere), and only in a homogeneous medium, light propagates in a straight line.

- A Ray of light is a line that shows the direction of propagation of a light wave.
- A Beam of light is a collection of light rays.

The Mathematics of Light
Ms. Gina Rablau, Kettering University
The Index of Refraction

- In **vacuum** all electromagnetic waves - including visible light - travel at the same speed:

\[ c = 2.99792458 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s} \]

- The **index of refraction** of a medium is defined as the ratio of the speed of light in vacuum, \( c \), and the speed of light in that medium, \( v \)

\[ n = \frac{c}{v} \]

- Except for vacuum, the index of refraction of any medium has a (slight) dependence on the color (i.e., on the wavelength \( \lambda \)) of the light – this is called **dispersion of light**: different colors propagate through the medium at different speeds

\[ n = n(\lambda) \rightarrow v = v(\lambda) \]
**Refraction of Light**

- Refraction of Light: “bending” of a light ray when it crosses the boundary (interface) between two different optical media.

- Snell’s Law
  - \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \)

- Snell’s Law Calculator
  [http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/refr.html#c1](http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/refr.html#c1)

- Refraction JAVA applet:
Refraction of Light – “Take 2”

- Refraction of Light: “bending” of a light ray when it crosses the boundary (interface) between two different optical media.

- Snell’s Law: \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \) (or \( n_i \sin \theta_i = n_t \sin \theta_t \))

- Consequences:
  - (A) If \( n_1 < n_2 \), then \( \theta_2 < \theta_1 \), i.e. the light ray is bent towards the normal
  - (B) If \( n_1 > n_2 \), then \( \theta_2 > \theta_1 \), i.e. the light ray is bent away from the normal
Total Internal Reflection

- When light is incident upon a medium of lesser index of refraction, the exit (refraction) angle is greater than the incident angle.

- As the incident angle increases, the exit angle increases as well; when the incident angle reaches a value called the critical angle, $\theta_c$, the refraction angle will be equal to $90^\circ$.

- For angles of incidence greater than the critical angle, there will be total internal reflection (i.e. no light will be transmitted into the 2nd medium).
Total Internal Reflection

- Snell’s law at critical angle, \( n_1 \sin \theta_c = n_2 \sin 90^\circ \), leads to:

  \[
  \sin \theta_c = \frac{n_2}{n_1} \quad \text{which means} \quad \theta_c = \arcsin \left( \frac{n_2}{n_1} \right)
  \]

- JAVA Applet: The critical angle of reflection
  
  http://micro.magnet.fsu.edu/primer/java/refraction/criticalangle/index.html

- JAVA Applet: Total internal reflection at water-air interface
  
  http://www.phy.ntnu.edu.tw/java/light/flashLight.html

Application: Optical Fibers

- The light is trapped in the (solid) core of the fiber and propagates through it by **total internal reflection**.

![Diagram of optical fiber with labels for refractive index and radius](image)

**Refractive Index, \( n \)**
- \( n_{\text{core}} \) in the core (\( \text{SiO}_2, \text{GeO}_2 \))
- \( n_{\text{cladding}} \) in the cladding (\( \text{SiO}_2 \))

**Radius**
- Core: 9 \( \mu \text{m} \)
- Cladding: 125 \( \mu \text{m} \)

---

The Mathematics of Light
Ms. Gina Rablau, Kettering University

Kettering University
Image Formation by Lenses

To create an image of an object, we use lenses and/or mirrors.
Lenses

- A *lens* is a transparent material (like glass or plastic) that (usually) has two *spherical* surfaces.

- *Converging lenses* are lenses that are thinner at the edges and thicker at the center. They transform a parallel beam of light into a converging beam of light (i.e. they “focus” together the light rays incident on the lens).
**Focal Points and Focal Length**

- **Object focal point**: the point where one has to place a point light source for the emerging beam to be a parallel beam of light.

- **Image Focal Point**: a parallel light beam parallel to the optical axis of the lens will be focused at a point called the image focal point.
The two focal points are located at the same distance $f$ from the lens, called the focal length of the lens.

The thin lens equation is an equation that relates (connects) the distance between the object and the lens ($s_o$) to the distance between its image and the lens ($s_i$)

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

Our goal is to use this equation to find where the image is formed (to find $s_i$) for various positions of the object (various object distances $s_o$).
JAVA Applets for Lenses

- Converging Lens Focal Point
  http://micro.magnet.fsu.edu/primer/java/components/perfectlens/index.html
- Simple Bi-Convex Thin Lenses
- Magnification with a Bi-Convex Lens
- Image Formation with Converging Lenses
- Image Formation with Diverging Lenses
- Geometrical Construction of Ray Diagrams – Thick Lenses
  http://micro.magnet.fsu.edu/primer/java/components/characteristicrays/index.html
A point light source is placed on the optical axis at a distance of 60 cm in front (i.e. to the left) of a converging lens that has a focal length of 10 cm. Calculate the distance between the lens and the image.
A point light source is placed on the optical axis at a distance of 60 cm in front (i.e. to the left) of a converging lens having a focal length of 10 cm. Calculate the distance between the lens and the image.

\[ f = 100 \text{mm} \]

\[
\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}
\]

\[
s_i = \frac{s_o \cdot f}{s_o - f} = \frac{(60) \cdot (10)}{60 - 10} = 12 \text{cm}
\]
Ray Tracing

➢ To construct graphically the image of an object, we use a drawing technique called **ray tracing**. Ray tracing makes use of three particular light rays:

1. A ray (any ray) passing through the **optical center** of the lens will pass undeviated (will continue in a straight line through the lens).
2. A ray (any ray) incident on the lens that is parallel to the optical axis will be refracted (bent) through the image focal point of the lens.
Ray Tracing

3. A ray (any ray) incident on the lens and passing through the **object focal point** of the lens will be refracted parallel to the optical axis.

![Converging (Convex) Lens Diagram]

- Object Focal Plane
- Optical center of the lens, O
- Image Focal Plane
- $f_0 = f$
- $f_i = f$

The Mathematics of Light
Ms. Gina Rablau, Kettering University
Ray Tracing

- The image of a point of the object is found at the intersection of these three light rays. In fact, any two of the three rays are enough for constructing the image of a point of the object.
- The image of the entire object is the totality of images of the constituent object points.

![Diagram of a converging (convex) lens with labeled parts: Object, Focal Plane, Image, Focal Plane, Object Focal Plane, Image Focal Plane, \( F_0 \), \( F_i \), \( f_0 = f \), \( f_i = f \).]
Image Formation Case Study 1

The Mathematics of Light
Ms. Gina Rablau, Kettering University
Image Formation Case Study 1
Image Formation Case Study 1

The Mathematics of Light
Ms. Gina Rablau, Kettering University
Image Formation Case Study 1

The Mathematics of Light
Ms. Gina Rablau, Kettering University
Image Formation Case Study 1

Diagram showing the optical properties of a bi-convex lens, including the object point, optical center, principal axis, focal points, and bi-convex lens.
Image Formation Case Study 1
Image Formation Case Study 1

The Mathematics of Light
Ms. Gina Rablau, Kettering University
Image Formation

- JAVA Applets: Image Formation through Converging Lenses

- JAVA Applet: Image Formation through Diverging Lenses