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Compound Interest, Annuities, Perpetuities and Geometric Series

Windows User

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Compound Interest, Annuities, Perpetuities and Geometric Series

A Motivating Example for Module 3

Project Description

This project demonstrates the following concepts in integral calculus:

1. Sequences.
2. Sum of a geometric progression.
3. Infinite series.

Project description.

Find the accumulated amount of an initial investment after certain number of periods if the interest is compounded every period. Find the future value (FV) of an annuity. Find the present value (PV) of an annuity and of a perpetuity.

Strategy for solution.

1. Obtain a formula for an accumulated amount of an initial investment after one, two, and three compounding periods. Generalize the formula to any number of periods.
2. Analyze the FV of an annuity using the results in step 1.
3. Analyze the PV of every annuity payment and consider the sum.
4. Perpetuity is a perpetual annuity; consider its PV as an infinite series.

1. We divide develop the general formula for the accumulated amount $A(n)$, or the future value of a payment P, at the end of the n-th period by first analyzing the first three periods. We assume the interest rate is i and the initial investment is P.

Period	Principal	Interest earned	Accumulated amount $A(n) = FV$
1	P	iP	$P+iP=P(1+i)$
2	$P(1+i)$	$iP(1+i)$	$P(1+i) + iP(1+i) = P(1+i)^2$
3	$P(1+i)^2$	$i\,P(1+i)^2$	$P(1+i)^2 + i\,P(1+i)^2 = P(1+i)^3$

The general formula is

$$FV = A(n) = P(1+i)^n$$

How does the accumulated value change with time?
How does the accumulation value change when the interest rate increases?

Numerical example. Compute the first three accumulated amounts for any selected values of P and i, perhaps you have some money in a savings account that pays interest, find out how your money will grow.

2. Imagine you are an investor wishing to accumulate certain amount $A = FV$ by making level payments for a certain number of periods. How much should the level payment be?

First, let's figure out the FV of n payments of \$1.

Here is the time diagram:

Payment		\$1	…	\$1	\$1	\$1
Period	0	1	…	n-2	n-1	n

The FV of the last payment made at time n is just \$1.

The FV of the next to last payment made at time n - 1 is just \$$(1 + \mathrm{i})$, the accumulated value of a payment of \$1 over one period.

The FV of the payment made at time *n - 2* is $\$(1+\mathrm{i})^2$, the accumulated value of a payment of $1 over two periods.

The FV of the payment made at time *1* is $\$(1+\mathrm{i})^{n-1}$, the accumulated value of a payment of $1 over *n-1* periods.

The FV of all payments is the sum

$$FV = 1 + (1+i) + (1+i)^2 + (1+i)^3 + \cdots + (1+i)^{n-1}$$
$$= \sum_{k=0}^{n-1} (1+i)^k$$

We need to find a closed formula for the sum of the geometric progression.

Here is the method, let $S = 1 + r + r^2 + \cdots + r^{n-1} = \sum_{k=0}^{n-1} r^k$

$$S = 1 + r + r^2 + \cdots + r^{n-1}$$
$$rS = \quad r + r^2 + \cdots + r^{n-1} + r^n$$

The difference is

$$S - rS = 1 - r^n$$
$$S(1-r) = 1 - r^n$$
$$S = \sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}$$

In our case *r = 1+i*, and we have

$$FV = \frac{(1+i)^n - 1}{i}$$

The FV of payments of $\$R$ *is*

$$FV = R\frac{(1+i)^n - 1}{i}$$

In order to accumulate the amount of *$A=FV*, the required payment is

$$R = A\frac{i}{(1+i)^n - 1}$$

Numerical example. Imagine you are an investor. Think how much you would like to accumulate and in how much time. You can now compute the amount of level payments you will have to make. To make this example more realistic, think of a car you would like to buy in 3 years. What's its estimated price in 3 years? This is the FV. What is the monthly interest rate in your bank? How much do you need to deposit every month to be able to buy this car?

3. First we need to understand the concept of PV, the present value.
   (a) How much money need to be put into an account that pays interest of i, to accumulate the amount of \$1 after one period? This will be the PV of \$1.

We need to solve the equation

$$PV(1+i) = 1$$

The solution is

$$PV = \frac{1}{1+i} = \nu$$

We use the symbol ν (small Greek letter "nu") to denote the fraction $\frac{1}{1+i}$. This quantity is called the discount factor.

   (b) How much money need to be put into an account that pays interest of i, to accumulate the amount of \$1 after two periods? This will be the PV of \$1, but two periods earlier.

We need to solve the equation

$$PV(1+i)^2 = 1$$

The solution is

$$PV = \frac{1}{(1+i)^2} = \nu^2$$

   (c) In general, how much money need to be put into an account that pays interest of i, to accumulate the amount of \$1 after n periods? This will be the PV of \$1, but n periods earlier.

$$PV = \frac{1}{(1+i)^n} = \nu^n$$

Now that we understand the concept of PV we can analyze annuities.

Annuity is a series of payments made at equal time intervals of time. Examples include house rents, mortgage payments, installment payments on cars, and interest payments on money invested. Consider an annuity under which payments of $1 are made at the end of each period for *n* periods.

Now assume you want to receive equal payments of $1 for *n* periods. How much should you invest in a fund that pays *i* compound interest?

Here is the time diagram:

Payment		$1	$1	$1	...	$1
Period	0	1	2	3	...	n

We need to calculate the sum of PV's of the above stream of payments. This will be the sum of individual PV's

$$PV = \nu + \nu^2 + \nu^3 + \cdots + \nu^n = \sum_{k=1}^{n} \nu^k = \nu \sum_{k=0}^{n-1} \nu^k$$

We need to find a closed formula for this sum. Using the general result on geometric progression with $r = \nu$

$$PV = \nu \frac{\nu^n - 1}{\nu - 1} = \frac{1 - \nu^n}{i}$$

as

$$\frac{\nu}{\nu - 1} = -\frac{1}{i}$$

So now we know how much needs to be invested to obtain n payments of $\$R$ in the form of an annuity

$$PV = R\frac{1-v^n}{i}$$

Numerical example. Imagine you are an investor. Think how many payments and of what amount you would like to receive in the future. Assume a certain interest rate. Then you are ready to compute how much you need to invest.

You can reverse this question. Think how much money you are willing to invest in an annuity. Think about the number of payments and the current interest rate. What would be your periodic payment?

4. Perpetuity is an annuity with payments continuing forever.

The perpetuity is not as abstract a concept as you may think. The British-issued bonds, called consols, is an example of a perpetuity. By purchasing a consol from the British government, the bondholder is entitled to receive annual interest payments forever.

Another example is a type of government bond called an undated issue that has no maturity date and pays interest in perpetuity.

An example of undated issues are the U.K. government's undated bonds or gilts, of which there are eight issues in existence, some of which date back to the 19th century. The largest of these issues presently is the War Loan, with an issue size of £1.9 billion and a coupon rate of 3.5% that was issued in the early 20th century.

Here is the time diagram of a perpetuity paying $1:

Payment		$1	$1	$1	…	$1	…
Period	0	1	2	3	…	n	…

We can compute its present value as an infinite series

$$PV = v + v^2 + v^3 + \cdots + v^n \ldots = \sum_{k=1}^{\infty} v^k$$

The sum is defined as

$$\sum_{k=1}^{\infty} \nu^k = \lim_{n\to\infty} \sum_{k=1}^{n} \nu^k = \lim_{n\to\infty} \frac{1-\nu^n}{i} = \frac{1}{i}$$

This is the amount that needs to be invested to receive payments of $1 forever, give the interest rate holds at *i* forever.

Numerical example. Imagine you want to support a charity. Think of an amount of periodic payment you would like to give forever to this charity. Assume a certain interest rate. Then you are ready to compute how much money you need to invest.