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The Electric Field of a Line of Charge

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The Electric Field of a Line of Charge

A Motivating Example for Modules 1 and 2

Project Description

This project demonstrates the following concepts in integral calculus:

PART 1 for Module 1

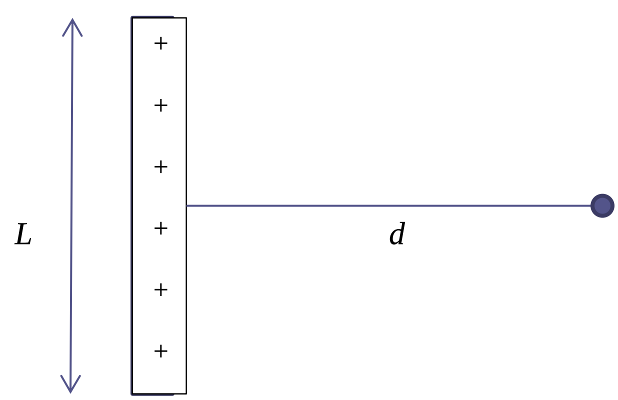
1. Riemann sums.
2. Indefinite integrals.
3. Definite integrals.

PART 2 for Module 2

4. Trigonometric substitution.
5. Trigonometric integrals.

Project description.

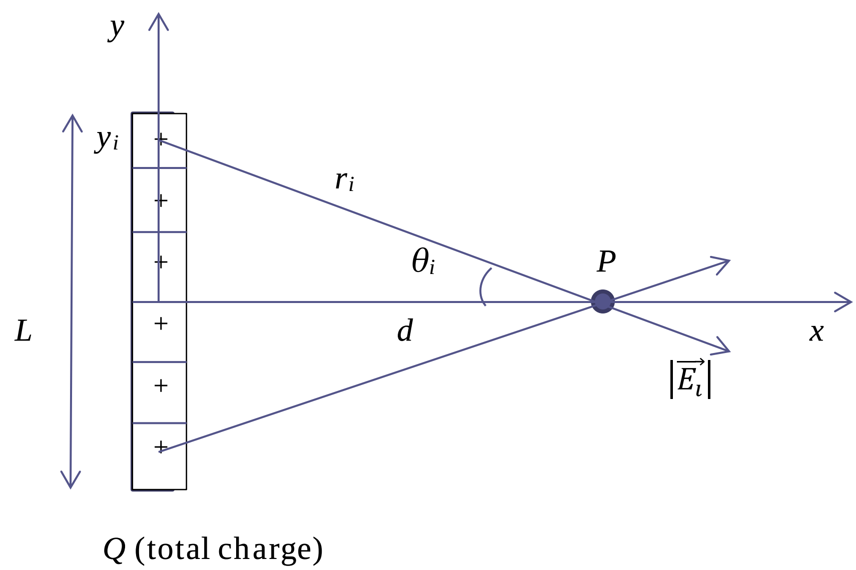
A thin, uniformly charged rod of length L with total charge Q generates electric field. Find its strength at distance *d* in the plane that bisects the rod.



Q (total charge)

Strategy for solution.

1. The total charge Q will be divided into many small point-like charges ΔQ.
2. Electric field from each of these point-like charges ΔQ will be determined.
3. The net field will be found by summing the fields of all the point-like charges ΔQ, forming a Riemann sum.
4. By taking the limit as the number of point-like charges ΔQ increases to infinity, the Riemann sum will converge to a definite integral.
5. The integral can be evaluated using trigonometric substitution and trigonometric integration.



1. We divide the rod into small segments. Each segment will be treated as a point charge. Since the vertical y component of the field generated by the upper half of the rod will cancel the vertical y component of the field generated by the lower half of the rod, only the x components need to be found to compute the net electric field.
2. The strength of electric field of a point charge q at a distance r is

$$|\vec{E}| = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2},$$

where $\varepsilon_0 = 8.85 \times 10^{-12} C^2/Nm^2$ is the permittivity constant (C is the abbreviation of coulomb, unit of charge, N is for newton, the unit of force, and m is for meters).

The distance between the i^{th} segment on the rod and point P is

$$r_i = \sqrt{y_i^2 + d^2},$$

where y_i is the vertical distance of the center if the i^{th} segment of the rod from the rod's center. Also

$$\cos(\theta) = \frac{d}{r_i} = \frac{d}{\sqrt{y_i^2 + d^2}}$$

Therefore, the x-component of $\overrightarrow{E_i}$, the electric field generated by segment i of the rod, is given by

$$\begin{aligned} E_i^x = E_i \cos(\theta) &= \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q}{r_i^2}\cos(\theta) \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q}{y_i^2 + d^2}\frac{d}{\sqrt{y_i^2 + d^2}} = \\ &= \frac{1}{4\pi\varepsilon_0}\frac{(d)(\Delta Q)}{(y_i^2 + d^2)^{3/2}} \end{aligned}$$

3. The net field is the sum of all the x-components for all segments,

$$E^x = \sum_{i=1}^{n} E_i^x = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{(d)(\Delta Q)}{(y_i^2 + d^2)^{3/2}}$$

The charge of each segment is the fraction of the total charge depending of the length of the segment Δy, that is, the charge of each segment is $Q(\Delta y/L) = (Q/L)\Delta y$. Hence, the net field is

$$E^x = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{(d)(Q/L)\Delta y}{(y_i^2 + d^2)^{3/2}} = \frac{(Q/L)(d)}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{\Delta y}{(y_i^2 + d^2)^{3/2}}$$

4. This is a Riemann sum, which as $n \to \infty$, converges to the following integral,

$$\frac{(Q/L)(d)}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{dy}{(y^2 + d^2)^{3/2}}$$

Here is evaluation of Riemann sums using MuPad for an 8.0 cm long rod.

```
f:=1/(x^2+8^2)^(3/2)
```

$$\frac{1}{(x^2+64)^{3/2}}$$

```
g:=plot::Function2d(f,x=-4..4)
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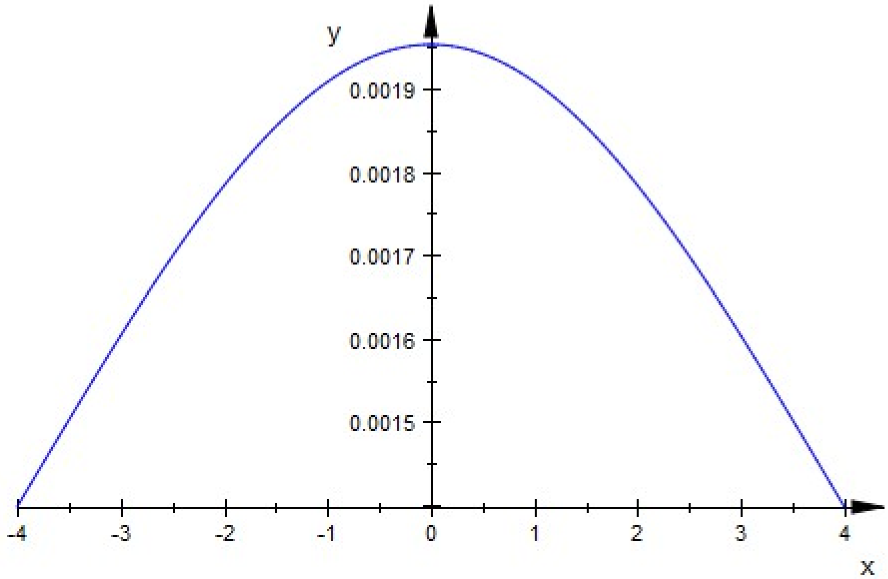
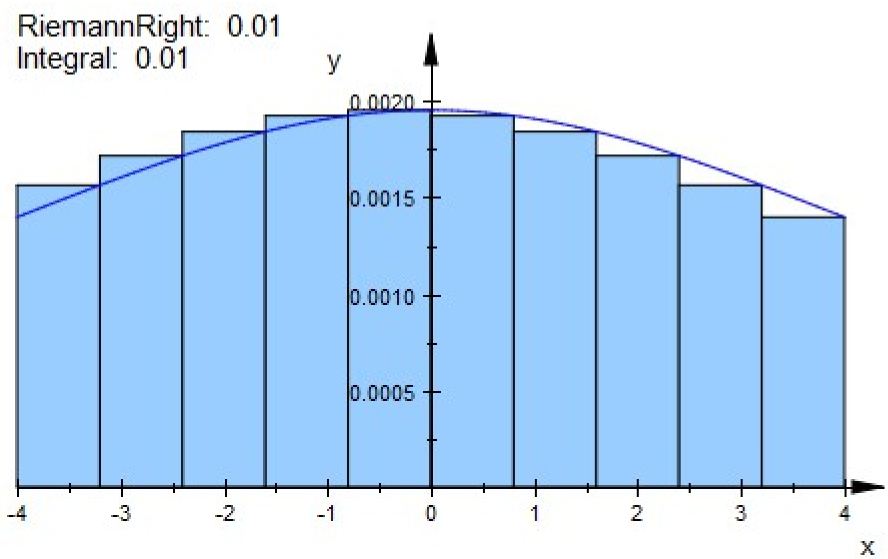
$$\text{plot::Function2d}\left(\frac{1}{(x^2+64)^{3/2}}, x = -4..4\right)$$

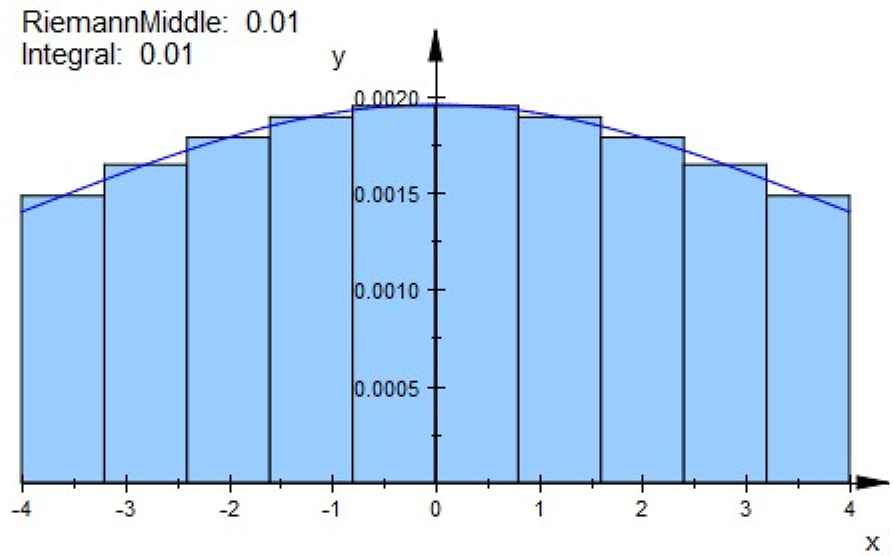
```
plot(g)
```




```
plot(plot::Integral(g,10,IntMethod=RiemannRight),f,x=-4..4)
```



```
plot(plot::Integral(g,10,IntMethod=RiemannMiddle),f,x=-4..4)
```




```
int(f,x=-4..4)
```

$\frac{\sqrt{5}}{160}$

```
float(%)
```

0.01397542486

The End of Part 1

Part 2

5. This kind of integral can be evaluated using trigonometric substitution.

$$y = d\,tan(\theta)$$
$$dy = d\,sec^2(\theta)d\theta$$

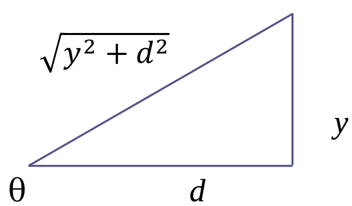
Using this substitution, we obtain

$$\begin{aligned}(y^2+d^2)^{3/2} &= [d^2tan^2(\theta)+d^2]^{3/2}\\ &= [d^2(tan^2(\theta)+1)]^{3/2} = d^3(sec^2\theta)^{3/2}\\ &= d^3sec^3\theta.\end{aligned}$$

Now we can evaluate the indefinite integral

$$\begin{aligned}\int\frac{dy}{(y^2+d^2)^{\frac{3}{2}}} &= \int\frac{d\,sec^2(\theta)d\theta}{d^3sec^3\theta}\\ &= \frac{1}{d^2}\int\frac{1}{\sec(\theta)}d\theta = \frac{1}{d^2}\int\cos(\theta)\,d\theta\\ &= \frac{1}{d^2}\sin(\theta)+C.\end{aligned}$$

Now we have to change the variable back from θ to y. Since $y = d\,tan(\theta)$, we use the following triangle,



From this triangle,

$$\sin(\theta) = \frac{y}{\sqrt{y^2+d^2}}$$

and we obtain that

$$\int\frac{dy}{(y^2+d^2)^{\frac{3}{2}}} = \frac{1}{d^2}\frac{y}{\sqrt{y^2+d^2}}+C$$

Now we need to evaluate the definite integral,

$$\frac{(Q/L)(d)}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{dy}{(y^2+d^2)^{3/2}}$$

$$=\left[\frac{(Q/L)(d)}{4\pi\varepsilon_0}\frac{1}{d^2}\frac{y}{\sqrt{y^2+d^2}}\right]_{-L/2}^{L/2}$$

$$=\frac{(Q/L)}{4\pi\varepsilon_0}\frac{1}{d}\left[\frac{L/2}{\sqrt{(L/2)^2+d^2}}-\frac{-L/2}{\sqrt{(L/2)^2+d^2}}\right]=\frac{1}{4\pi\varepsilon_0}\frac{Q}{d\sqrt{(L/2)^2+d^2}}$$

To allow the possibility that the charge can be negative, the final solution will use absolute value of Q,

$$E^x=\frac{1}{4\pi\varepsilon_0}\frac{|Q|}{d\sqrt{(L/2)^2+d^2}}.$$

Numerical Example

We find the electric field strength 1.0 cm from the middle of an 8.0 cm long glass rod that has been charged to 10 nC.

$L = 0.08\ m$, $d = 0.01\ m$, $Q = 10^{-8}\,C$

$$E^x=\frac{1}{4\pi\varepsilon_0}\frac{|Q|}{d\sqrt{(L/2)^2+d^2}}$$

$$=\frac{1}{4\pi(8.85\times10^{-12})}\frac{|10^{-8}|}{0.01\sqrt{(0.08/2)^2+0.01^2}}$$

$$=2.2\times10^5\ N/C$$