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The Riemann Zeta Function – Unsolved Mystery

Leszek Gawarecki

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The Riemann Zeta Function – Unsolved Mystery

A Motivating Example for Module 3

Project Description

This project demonstrates the following concepts in integral calculus:

1. Infinite series.
2. p-series.
3. Integral test for convergence/divergence.

Project description.

Examine convergence/divergence of p-series. Learn about the Riemann zeta function and some of its specific values.

Strategy for solution.

1. Apply the integral test for convergence/divergence of infinite series.

1. A p-series is an infinite series and it defines the Riemann zeta function for the values of $p \geq 1$,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

2. The Integral Test. Suppose f is a continuous, positive, and decreasing function defined on $[1, \infty)$, and $f(n) = a_n$. Then
   (a) The series $\sum_{n=1}^{\infty} a_n$ converges if the improper integral $\int_1^{\infty} f(x)dx$ converges
   (b) The series $\sum_{n=1}^{\infty} a_n$ diverges if the improper integral $\int_1^{\infty} f(x)dx$ diverges.

   Since the integral $\int_1^{\infty} \frac{1}{x^p} dx$ converges for $p > 1$ and diverges for $0 \leq p \leq 1$, the p-series converges for $p > 1$ and diverges for $0 \leq p \leq 1$, and the Riemann zeta function has finite values for $p > 1$.

3. Specific Values of the Riemann zeta function.
   (a) $p = 1$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

   is the divergent harmonic series.

   (b) $p = 3/2$

$$\zeta(3/2) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \frac{1}{4^{3/2}} + \cdots$$
$$\approx 2.6124$$

   This constant is present in the formula for the critical temperature at which a transition to a Bose-Einstein condensate occurs. A Bose-Einstein condensate is a state of matter of a gas of bosons cooled to temperatures very close to absolute zero (0 K or -273.15 C). Under such conditions quantum phenomena can be observed on macroscopic scale.

$$T_c = \left(\frac{\rho_N}{\zeta(3/2)}\right)^{2/3} \frac{2\pi\hbar^2}{mk_B}$$

   T_c is the critical temperature
   ρ_N is the particle density
   m is the mass per boson
   $\hbar$ is the reduced Planck constant
   k_B is the Boltzmann constant

   This state was predicted in 1924-25 by Satyendra Nath Bose and Albert Einstein.

(c) $p = 2$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

(d) $p = 3$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.202$$

This is called the Apéry constant, which appears in the formula for gyromagnetic ratio. This has application in Nuclear Magnetic Resonance and Magnetic Resonance Imaging.

(e) $p = 4$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

This constant appears in Stefan-Boltzmann law (which describes the power radiated from a black body in terms of its temperature).and Wien approximation.

4. Relation to prime numbers.

A positive number $q > 1$ is prime if it is divisible only by 1 and by itself. For example $2, 3, 5, 7, 11$ are prime numbers.

The following is called the Euler product formula,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \prod_{all\ q\ prime} \frac{1}{1 - q^{-p}}$$

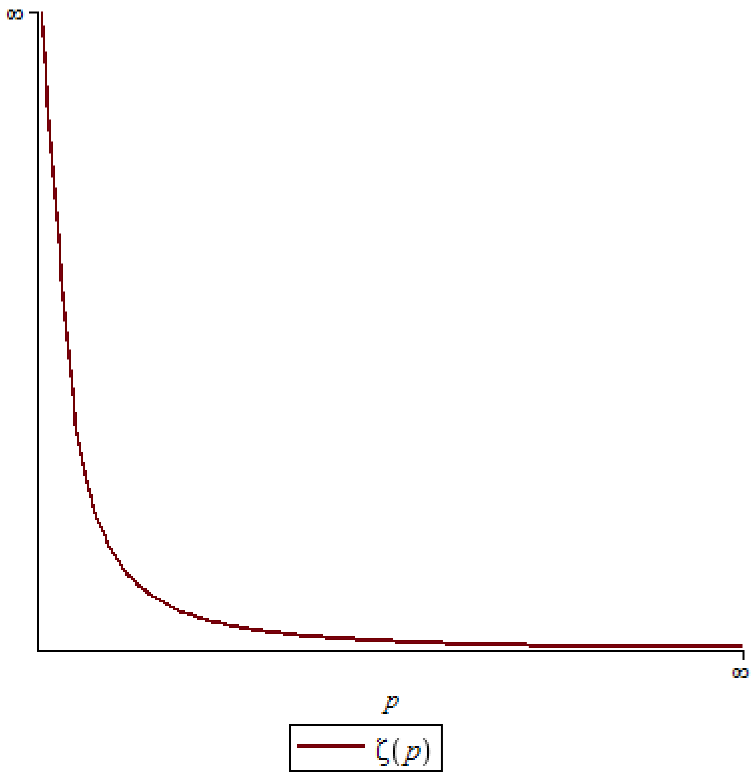
For example,

$$\left(\frac{1}{1-2^3}\right)\left(\frac{1}{1-3^3}\right)\left(\frac{1}{1-5^3}\right)\left(\frac{1}{1-7^3}\right)\left(\frac{1}{1-11^3}\right) \approx 1.200784628 \approx 1.202 \approx \zeta(3)$$

Intuitively, chances that any number is divisible by a prime q are $\frac{1}{q}$. Chances that k numbers are divisible by a prime q are $\left(\frac{1}{q}\right)^k$. For distinct primes, the divisibility events are mutually independent, since a number is divisible by two primes q_1 and q_2 if and only if it is divisible by the product $q_1 q_2$, which happens with probability $\frac{1}{q_1 q_2}$. Thus the probability that k numbers are co-prime (meaning they have no common divisors other than 1) is given by a product over all primes

$$\prod_{all\ q\ prime} \frac{1}{1-q^{-k}} = \left(\prod_{all\ q\ prime} \frac{1}{1-q^{-k}} \right)^{-1} = \frac{1}{\zeta(k)}$$

5. The graph of $\zeta(p)$.



6. The unsolved mystery.

The Riemann zeta function can be defined for complex numbers, $\zeta(z)$. It is known that $\zeta(z) = 0$ for $z = -2, -4, -6, \ldots$ negative even integers. The Riemann hypothesis, stated in 1859 by Bernhard Riemann, says that $\zeta(z) = 0$ also only for $z = \frac{1}{2} + it$, where i is the imaginary unit. This statement has not been yet proven despite many attempts.