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Work Done by a Force in Moving an Object

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Work Done by a Force in Moving an Object

A Motivating Example for Module 1

Project Description

This project demonstrates the following concepts in integral calculus:

1. Riemann Sums
2. Definite Integrals

Work

In this project we will be looking at the amount of work that is done by a force in moving an object.

In a first course in Physics you typically look at the work that a constant force, *F*, does when moving an object over a distance of *d*. In these cases the work is,

$$W = Fd$$

However, most forces are not constant and will depend upon where exactly the force is acting. So, let's suppose that the force at any *x* is given by *F(x)*. Then the work done by the force in moving an object from to x = a to x = b is given by,

$$W = \int_a^b F(x)\,dx$$

Strategy for solution.

1. The object will be divided into many pieces of equal width Δx.
2. We will approximate each piece by a cylinder.
3. An approximation to the total work done to move the entire object will be found by adding the work to move each piece of the object – Riemann sum
4. By taking the limit as the number of pieces increases to infinity, the Riemann sum will converge to a definite integral.

Numerical Example

A tank in the shape of an inverted cone has a height of 15 meters and a base radius of 4 meters and is filled with water to a depth of 12 meters. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is 1000 kg/m^3.

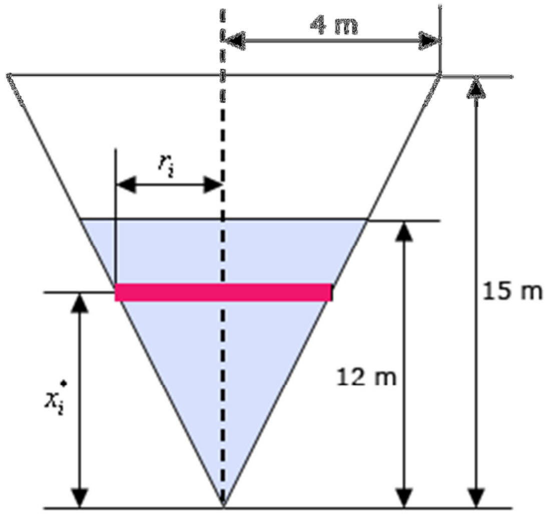
Solution

In this case we cannot just determine a force function, F(x) that will work for us. So, we are going to need to approach this from a different standpoint.

Let's first set x = 0 to be the lower end of the tank/cone and x = 15 to be the top of the tank/cone. With this *definition* of our *x*'s we can now see that the water in the tank will correspond to the interval [0, 12]

So, let's start off by dividing [0, 12] into *n* subintervals each of width Δx and let's also let x_i be any point from the i^{th} subinterval where i = 1, 2,…n. Now, for each subinterval we will approximate the water in the tank corresponding to that interval as a cylinder of radius r_i and height Δx .

Here is a quick sketch of the tank. Note that the sketch really isn't to scale and we are looking at the tank from directly in front so we can see all the various quantities that we need to work with.



The red strip in the sketch represents the "cylinder" of water in the i^{th} subinterval. A quick application of similar triangles will allow us to relate r_i to x_i (which we'll need in a bit) as follows

$$\frac{r_i}{x_i^*} = \frac{4}{15} \qquad \Rightarrow \qquad r_i = \frac{4}{15} x_i^*$$

The mass, m_i of the volume of water, V_i, for the i^{th} subinterval is simply,

$$m_i = \text{density} \times V_i$$

We know the density of the water (it was given in the problem statement) and because we are approximating the water in the i^{th} subinterval as a cylinder we can easily approximate the volume,

$$V_i \approx \pi (\text{radius})^2 (\text{height})$$

and hence the mass of the water in the i^{th} subinterval.

The mass for the i^{th} subinterval is approximately,

$$m_i \approx (1000)\left[\pi r_i^2 \Delta x\right] = 1000\pi \left(\tfrac{4}{15} x_i^*\right)^2 \Delta x = \tfrac{640}{9}\pi \left(x_i^*\right)^2 \Delta x$$

To raise this volume of water we need to overcome the force of gravity that is acting on the volume and that is, $F = m_i g$ where $g = 9.8 \text{ m/s}^2$ is the gravitational acceleration. The force to raise the volume of water in the i^{th} subinterval is then approximately,

$$F_i = m_i g \approx (9.8)\tfrac{640}{9}\pi\left(x_i^*\right)^2 \Delta x$$

Next, in order to reach to the top of the tank the water in the i^{th} subinterval will need to travel approximately $15 - x_i$ to reach the top of the tank. Because the volume of the water in the i^{th} subinterval is constant the force needed to raise the water through any distance is also a constant force.

Therefore the work to move the volume of water in the i^{th} subinterval to the top of the tank, *i.e.* raise it a distance of $15 - x_i$ is then approximately,

$$W_i \approx F_i\left(15 - x_i^*\right) = (9.8)\tfrac{640}{9}\pi\left(x_i^*\right)^2\left(15 - x_i^*\right)\Delta x$$

The total amount of work required to raise all the water to the top of the tank is then approximately the sum of each of the W_i for i = 1, 2, …n. Or,

$$W \approx \sum_{i=1}^{n}(9.8)\tfrac{640}{9}\pi\left(x_i^*\right)^2\left(15 - x_i^*\right)\Delta x$$

To get the actual amount of work we simply need to take n -> ∞. *I.e.* compute the following limit,

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}(9.8)\tfrac{640}{9}\pi\left(x_i^*\right)^2\left(15 - x_i^*\right)\Delta x$$

Recalling the definition of the definite integral we can see that this is nothing more than the following definite integral,

$$W = \int_0^{12}(9.8)\tfrac{640}{9}\pi x^2(15 - x)\,dx = (9.8)\tfrac{640}{9}\pi\int_0^{12}15x^2 - x^3\,dx$$

$$= (9.8)\tfrac{640}{9}\pi\left(5x^3 - \tfrac{1}{4}x^4\right)\Big|_0^{12} = 7{,}566{,}362.543\text{ J}$$